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QUADRATIC VOLUME PRESERVING MAPS: AN EXTENSION OF A RESULT OF MOSER

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A natural generalization of the Hénon map of the plane is a quadratic diffeomorphism that has a quadratic inverse. We study the case when these maps are volume preserving, which generalizes the the family of symplectic quadratic maps studied by Moser. In this paper we obtain a characterization of tliese maps for dimension four and less. In addition, we use Moser's result to construct a subfamily of in n dimensions.

1. Introduction

Some of the simplest nonlinear systems are given by quadratic maps: for example the logistic map in one dimension and the quadratic map introduced by Hénon [14, 15] in the plane. It is easy to see that any quadratic, one dimensional map with a fixed point is affinely conjugate to the logistic map, *xy-^ rx{\ — x).* In a similar way, Hénon showed that a generic quadratic area-preserving mapping of the plane can be written in normal form as

$$:) - (\begin{matrix} k + y + x^ \\ -x \end{matrix}$$

which has a single parameter k.

Hénon's study can be generalized in several directions. Moser [22] studied a class of quadratic symplectic maps, having obtained a useful decomposition and normal form. For example, when the map is quadratic and symplectic in M^", Moser [22,19] showed that it can be written as the composition of twon in 5549 in o49 Tw0.127 Tc(dimensiona) 4(a) Tj1 where W is a homogeneous cubic polynomial in *p*. The map given in (1) is a particular example of what we call a quadratic shear.

Definition 1. A quadratic shear is a bijective map of the form

X ^ *fix)* = X + *-Qix),* (2)

where *Q{x)* is a vector of homogeneous, quadratic polynomials such that *f~^* is also a quadratic map.

In this way Moser's result is basically a characterization of all symplectic quadratic shears. One of the remarkable aspects of this is that quadratic symplectic maps necessarily have quadratic inverses. In general we can write a quadratic map on E" as the composition of an affine map with a quadratic map that is zero at the origin and is the identity at linear order:

x ^¥ *fix)* = *xo* + *L{x* +-*Q{x)),* (3)

where SQ S M", L is a matrix, and *Qix)* is a vector of homogeneous, quadratic polynomials. Note that if the map / is volume preserving then it is necessary that *L* satisfies det(L) = 1. Similarly if / is symplectic, then *L* must be a symplectic matrix. Of course, the quadratic terms also can not be chosen arbitrarily in these cases.

Polynomial maps are of interest from a mathematical perspective. Much work has been done on the "Cremona maps", that is polynomial maps with constant Jacobians [8]. An interesting mathematical problem concerning such maps maps",

ii)=>i) By assumption, det(D/(a;)) and det(D/ ^(/(a;))) are polynomials in *xi,X2*,-.- ,a:„. However, differentiation of *f~^(f{x))* = *x* gives

det{Dr\f{x)))detiDf{x))=^l,

and therefore, since both are polynomials, *det{Df{x))* has to be a constant independent of *x*. We notice that *det{Df{x))* = det{D/(0)) =

We will see that for the

A map / is symplectic with respect to w if *u}{Dfv,Dfv')* = *u{v,v')* for all

4. Dimensions Three and Four

Following Coroliary 1, we would like to establish the stronger result that M(a;)^ = 0 for all *x*. In this section x.

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