(b)   Using 2×2 matrices, construct an example where the product of two real symmetric matrices does not have real eigenvalues.

Problem 4.   Interpolation / Approximation

Let function $f \in C^{n+1}[a,b]$, $|f^{(n+1)}(x)| \le M$ and $E_n(f)$ be the error of its best approximation by a polynomial of degree n. Show that the accuracy of the best polynomial approximation improves rapidly as the size of the interval $[a,b]$ shrinks, i.e., show that

$$E_n(f) \le \frac{2M}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}.$$

Hint: Use the Chebyshev nodes $x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)\cos\frac{\pi}{2}\frac{2+1}{n+1}$ to construct a polynomial approximation of f.

Problem 5.   Numerical ODE