

Pulse Bifurcations in Stochastic Neural Fields*

Zachary P. Kilpatrick[†] and Grégory Faye[‡]

Abstract.

ff . N ! fi ! . . . ff . . . fi !
y ! y ! . . . ff ! ! . . . fi
y ! ! y . . . ff ! ! . . . fi
-

$\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$ is the inner product in $L^2(\mathbb{R}^d)$. For $\phi \in L^2(\mathbb{R}^d)$, we define the adjoint operator \mathcal{L}^* by

$$\langle \mathcal{L}\phi, \psi \rangle_{L^2(\mathbb{R}^d)} = \langle \phi, \mathcal{L}^*\psi \rangle_{L^2(\mathbb{R}^d)}$$
 for all $\phi, \psi \in L^2(\mathbb{R}^d)$. The adjoint operator \mathcal{L}^* is given by

$$\mathcal{L}^*\psi = -\operatorname{div}(\mathbf{D}\psi) + \mathbf{b} \cdot \nabla \psi + c\psi,$$
 where $\mathbf{D} = (D_{ij})_{i,j=1,\dots,d}$ is the diffusion matrix, $\mathbf{b} = (b_1, \dots, b_d)$ is the drift vector, and c is the reaction term. The adjoint operator \mathcal{L}^* is self-adjoint if and only if \mathbf{D} is symmetric and $\mathbf{b} = \nabla c$.

As $\gamma \rightarrow \infty$, $\theta \rightarrow 0$, and $\beta \rightarrow \infty$.

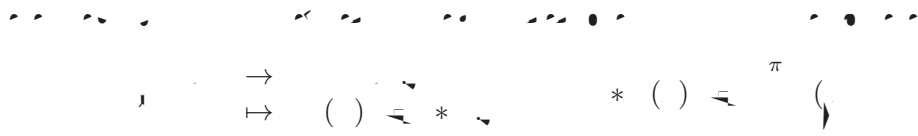
$$(1.1) \quad \frac{1}{\beta} \frac{d}{dt} \left(\frac{1}{\theta} \right) \leq 0 \quad \text{and} \quad \frac{1}{\theta} \geq \frac{1}{\theta_0}$$

By (1.1), $\frac{1}{\theta}$ is non-increasing and bounded below by $\frac{1}{\theta_0}$. Thus, θ is bounded above and $\theta \rightarrow \theta_\infty > 0$ as $t \rightarrow \infty$. Moreover, $\frac{1}{\beta} \frac{d}{dt} \left(\frac{1}{\theta} \right) \leq 0$ implies $\frac{d}{dt} \left(\frac{1}{\theta} \right) \leq 0$, and hence $\frac{1}{\theta}$ is non-increasing. This implies that θ is bounded above and $\theta \rightarrow \theta_\infty > 0$ as $t \rightarrow \infty$. Moreover, $\frac{1}{\beta} \frac{d}{dt} \left(\frac{1}{\theta} \right) \leq 0$ implies $\frac{d}{dt} \left(\frac{1}{\theta} \right) \leq 0$, and hence $\frac{1}{\theta}$ is non-increasing. This implies that θ is bounded above and $\theta \rightarrow \theta_\infty > 0$ as $t \rightarrow \infty$.

$$\frac{d}{dt} \left(\frac{1}{\theta} \right) \leq -\frac{1}{\beta} \frac{d}{dt} \left(\frac{1}{\theta} \right) + \frac{\pi}{-2\pi} \left(\frac{1}{\theta} \right)' \left(\frac{1}{\theta} \right) - \frac{1}{\theta}$$

$$(1.2) \quad \frac{d}{dt} \left(\frac{1}{\theta} \right) \leq \frac{\pi}{-2\pi} \left(\frac{1}{\theta} \right)' \left(\frac{1}{\theta} \right) - \frac{1}{\theta} + \alpha \left(\frac{1}{\theta} \right)$$

By (1.2), $\frac{d}{dt} \left(\frac{1}{\theta} \right) \leq \frac{\pi}{-2\pi} \left(\frac{1}{\theta} \right)' \left(\frac{1}{\theta} \right) - \frac{1}{\theta} + \alpha \left(\frac{1}{\theta} \right)$. This implies that $\frac{1}{\theta}$ is bounded above and $\frac{1}{\theta} \rightarrow \frac{1}{\theta_\infty}$ as $t \rightarrow \infty$.



(.) $\left(\frac{N}{k} \right) k (\cdot), \dots (\cdot)$

$k \frac{k}{+\beta} \pi (\cdot) \frac{N}{k} k (\cdot)$

(.) $(/) +$

fi (\cdot)



Figure 1. (A) i i i b c b i i a b i
 b c i b $($ $)$ $)$ i i i b c i i b $($ $)$ i c i i c i
 b c i i $pitch$ B i i i i b c i $($ SN $)$ i c i b c i
 cc i i $= 0.5$ c i i b b c i i SN $)$ i i i i b c i
 (B) i c i i b i b i i i $($ c $)$ b c i i i i i
 $B = 2$ $= 0.25$

Downloaded 05/23/14 to 130.49.198.5. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

$$\begin{aligned}
 & \dots (\dots) \dots (\dots) \dots \text{fi} \dots \\
 & \lambda = 0, \dots (\dots) \dots, \lambda = \beta / \alpha \dots \text{fi} \dots \\
 & \beta = \alpha. \dots (\dots) \dots \text{fi} \dots \\
 & \dots \beta, \dots (\dots) \dots \text{fi} \dots \\
 & (\dots) \dots (\dots) \dots \\
 & \dots (\dots) \dots
 \end{aligned}$$

... ζ ... ζ ...

$$/ \alpha \quad \alpha / \frac{1}{\alpha}$$

... fi ...

$$/ (+ \alpha) () + \int_{-\pi}^{\pi} (/) '(()) () \quad \frac{1}{\alpha}$$

... Ω ... 0 ... Ω ... Ω ... Ω ... 0 ... 0 ... 0 ... Ω

$(\zeta, \zeta) \in c_2$... fi ... s ... c .
 (\cdot, \cdot) ... Δ ... ζ ... $\Psi(\mu)$...

$$(\cdot, \cdot) \quad \Delta (\cdot + \zeta + \Psi(\mu))$$

... $\delta \in \mathbb{R}$

$$f'(\pm\alpha) + \frac{\pi}{-\pi} f'(\pm\alpha) = \frac{f''(\pm\alpha)}{\alpha}$$

... , ...

$$\frac{\Psi}{\zeta^*} x \sim \delta \frac{\zeta}{\zeta^*} x \sim \frac{1}{\alpha} \dots \sim \frac{f''}{\alpha f'}$$

$$f'' \sim \frac{\pi}{-\pi} f'(\pm\alpha) f''(\pm\alpha)$$

$$(\dots) \sim \frac{f''}{\alpha f'}$$

... f'' ...

$$f''(\pm\alpha) = \frac{\pi}{-\pi} f'(\pm\alpha) f''(\pm\alpha)$$

$$f'' \sim \frac{\pi}{-\pi} f'(\pm\alpha) f''(\pm\alpha) \sim \frac{\pi}{-\pi} f''(\pm\alpha)$$

$$\sim \frac{\pi}{-\pi} f''(\pm\alpha)$$

$$\sim \frac{\pi}{+\alpha} \frac{\pi}{-\pi} \frac{\pi}{-\pi} f'(\pm\alpha) f''(\pm\alpha) = 0$$

... (\dots) ... α, β ... (\dots) ...

$$(\dots) \sim \pm \frac{\alpha(\beta/\alpha) f''}{f''}$$

... f'' ...

$$\pm(\dots) \sim (\pm) + (\mu)$$

$$\pm(\dots) \sim (\pm) / \frac{\pm}{\alpha} f'(\pm\alpha) + (\mu)$$

¹N ... (1.2), ... = ... , || \cdot $\|_F^2 = \langle \dots \rangle$... $\pm = \pm\sqrt{\alpha(-\alpha)}$. A ... 3.1, ...

Let \mathbf{u}^\pm be the stable and unstable manifolds of $(\xi, \pm\alpha)$. Then $\mathbf{u}^\pm \approx (\xi, \pm\alpha)$ for $\xi \approx \pm\alpha$.

2.3. Perturbed amplitude equations at the drift bifurcation.

Let $(\cdot, 0)$ be the bifurcation point. For $\mu \approx \beta/\alpha > 0$, we have $\epsilon \approx \mu$, $\epsilon \approx \mu$, 0 . The bifurcation diagram shows $(\cdot, 0)$ and (\cdot, μ) . The bifurcation diagram shows $(\cdot, 0)$ and (\cdot, μ) .

Let $(\cdot, 0)$, $\Delta(\cdot) \approx \bar{\mu}$. The bifurcation diagram shows $(\cdot, 0)$ and (\cdot, μ) .

$$(\cdot, \cdot) \quad \bar{\mu}(\cdot) \approx (\cdot) / \beta(\cdot) + \frac{\pi}{2} (\cdot) ((\cdot))$$

$$(\cdot, \cdot) \quad \bar{\mu}(\cdot) \approx \alpha((\cdot)) / (\cdot) + \mu^{\frac{3}{2}} (\cdot)$$

Let (\cdot, \cdot) , $\bar{\mu}(\cdot, \mu^-)$. The bifurcation diagram shows (\cdot, μ^-) and (\cdot, μ) .

$$(\cdot, \cdot) \quad (\cdot) \approx (\cdot / \Delta(\cdot)) + \mu (x / \Delta(\cdot)) + \mu^{\frac{3}{2}} (\cdot / \Delta(\cdot)) + (\mu)$$

$$(\cdot, \cdot) \quad (\cdot) \approx (\cdot / \Delta(\cdot)) / \bar{\mu}(\cdot)$$

$$\begin{aligned}
 & \Delta \dots (\cdot, Q), \dots \\
 & \Delta \dots (\cdot), \dots \\
 & (\cdot) \sim \bar{\mu}(\mu) \dots \mathbb{E} \dots \\
 & (\cdot) = (\beta / \alpha) (\cdot) + \dots
 \end{aligned}$$

Downloaded 05/23/14 to 130.49.198.5. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

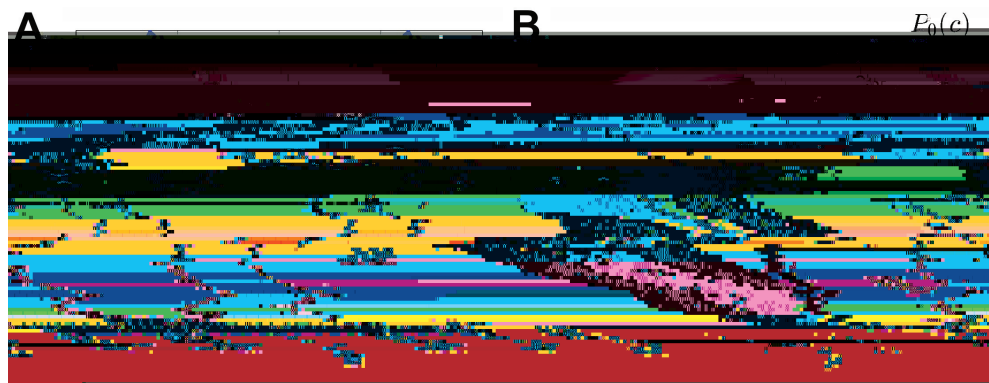


Figure 2. \mathcal{G}_1 is a function of α and c defined by (1.2), \mathcal{G}_2 is a function of α and c defined by (1.4), \mathcal{G}_3 is a function of α and c defined by (1.7). $\mathcal{C}(c) = \mathcal{G}_1(c)$. (A) $\mathcal{C}(c)$

$$\begin{aligned}
 & \dots \\
 & \dots \left(\left(\cdot \right) \right) \approx \int_{-\pi}^{\pi} \left(\cdot \right)' \left(\dots + \dots \right) \dots \\
 & \dots \left(\cdot \right), \dots \left(\cdot \right), \dots \left(\cdot \right). \\
 & \dots \text{fi} \dots \left(\cdot \right), \dots \\
 & \dots \text{fi} \dots \theta \quad \xi \dots \\
 & \dots \left(\cdot \right) \dots \\
 & \left(\cdot \right) \dots
 \end{aligned}$$

$\gamma_{\pm}(\xi)$

$$\gamma_{+} + \gamma_{-}$$

Downloaded 05/23/14 to 130.49.198.5. Redistribution subject to SIAM license or copyright; see <http://www.siam.org/journals/ojsa.php>

- [7] D. Blömker, M. Hairer, and G. A. Pavliotis, *Stochastic averaging for multiscale stochastic differential equations*, *Nonlinearity*, 20 (2007), . 1721–1744.
- [8] D. Blömker, S. Maier-Paape, and G. Schneider, *Stochastic averaging for multiscale stochastic differential equations*, *Discrete Contin. Dynam. Systems*, 1 (2001), . 527–541.
- [9] M. Bode,

- [33] C. W. Gardiner, *Stochastic Processes: With Applications*, John Wiley & Sons, 2009.

- [59] M. Tsodyks, K. Pawelzik, and H. Markram, *Nature*, 10 (1998), . 821–835.
- [60] R. Veltz and O. Faugeras, *Journal of the Royal Society B*, 9 (2010), . 954–998.
- [61] X.-J. Wang, *Nature*, 90 (2010), . 1195–1268.
- [62] H. R. Wilson and J. D. Cowan, *Biological Cybernetics*, 13 (1973), . 55–80.
- [63] W. Xu, X. Huang, K. Takagaki, and J.-Y. Wu, *Chaos*, 55 (2007), . 119–129.
- [64] K. Yoon, M. A. Buice, C. Barry, R. Hayman, N. Burgess, and I. R. Fiete, *Science*, 16 (2013), . 1077–1084.
- [65] L. C. York and M. C. W. van Rossum, *Chaos*, 27 (2009), . 607–620.