A draft of INRIA lectures, May 1991

Wavelets, Multiresolution Analysis and Fast Numerical Algorithms

Bey n

M po e o eq e o de N ope on o co p e e

$$p_j = \sum_{i \neq j} \frac{q_i q_j}{| \quad i|}$$

e e e od y e e ed de ce fo ed c n p d en eq on o
p e ne y e fo e co of n n e en y cond on n e of e e n
ce f n e d of n e d ence o n e e e en ep e en on e e e ep
e en on of e de e n e e e en p e od c on

II.1 Multiresolution analysis.

e e de n on of e e o on n y no on n od ced y Meye nd M c p e e en f ee n en e of e en o o of ene y e ; o e ne e n e d of e en e

$$\mathbf{V_n} \subset \subset \mathbf{V} \subset \mathbf{V} \subset \mathbf{V} \quad \mathbf{V} \subset \mathbf{L} \ \mathbf{R^d}$$

n n e c e z on e p ce **V** n e d en on

II.2 The Haar basis

o on e e on y e p e of e e o on n y fy n De n on Cond on e Co p n n o p e of e o e de e n e ec e nd p o de f p o o ype fo n e c e pe en on f d en $_{j;k}$ $^{j=}$ ^{j} $\in$ **Z** fo ed y e d on nd n on of n e f nc on

fo
fo $\leq$
e e e

n c e e e e c c e c f nc on of e n e o e c $_{j;k}$ $^{j=}$ ^{j} $\in$ **Z** e of $\mathbf{V_j}$ nd $_{j;k}$ $^{j=}$ ^{j} $\in$ **Z** e of $\mathbf{W_j}$

e deco po on of f nc on n o n o de N p oced e en N ^{n} p e of f nc on c y fo p c y e o of e of ed e e of f on n e of en ^{n}

$$_{k} = {}^{n=} \int_{-n\,k}^{-n\,k+\,)} f \; d$$

e o n coe c en

$$d_{k}^{j+} = \frac{}{\sqrt{}} \; {}^{j}_{k} - {}^{j}_{k}$$

nd e e

$$_{k}^{j+} = \frac{}{\sqrt{}} \; {}^{j}_{k} \quad {}^{j}_{k}$$

fo ; $n-$ nd ^{n} ^{j} $-$ e y o e e n e o e of coe c en d^{j}_{k} $^{j}_{k}$ n eq e $N-$ dd on nd N p c on n o d en on e e e o n o e nd d fo e econd de ned y e e of ee nd of f nc on ppo ed on q e j;k j;k′ y j;k j;k′ y nd j;k j;k′ y e e e c c e c f nc on of e n e nd j;k = j= j — ep e en n n ope o n e d o e non nd d fo e e no o y eco e c e e

By con de n n n e ope o

$$f \;=\; \int y \; f \; y \; dy$$

nd e p nd n e ne n o d en on [redacted] e nd fo C de on Zy nd nd p e do d en ope o e dec y of en e f nc on of e d nce f o e d on f e n e e ep e en on n n e o n e ne e e c of ope o e en y n e o d on e ne e oo y f o e d on o e p e e ne y of C de on Zy nd ope o fy e e e

$$| \quad y | \leq \frac{}{| \; - y|}$$

$$| \partial_x^M \quad y | \quad | \partial_y^M \quad y | \leq \frac{C_M}{| \; - y|^{\;+M}}$$

fo o e $M \geq$ Le M = n nd con de

$$\int\int y \;\; j;k \;\; j;k′ \; y \; d \; dy$$

e e e e e d nce e een $| \; - \; ′| \geq$ nce

$$\int j;k \; d \; =$$

e e

$$| \quad y \quad f \; r \quad \times$$

e on e n n e dec y n c en o e co p n n p c c o e f e dec y nece y o e f nc on e e n n o en e n n o en e e pon e fo n n p c c o e con o n e con n n e co p e y e e of e f o

II.3 Orthonormal bases of compactly supported wavelets

e q e on of e e ence of e on n y oo f nc on of Cond on e no e ed n e con c on of e o ono e e ene z n f nc on y o e nd Meye e con de on y co p c y ppo ed e e n n o en con c ed y D ec e fo o n e o of Y Meye nd M o o of e e e n d fo o e c o ce of e e e e f e e c o c o ce of e e y on y con de n fo d en on $d \geq$ e con c ed f o o e fo d.

Le con de e e on n y fo L R n f d d e

econd e o o on y of { − }k Z p e

$$\mathbf{k} \;-\; \int_{+} \;-\; d \;-\; \int_{+} |\;|\; e^{i\mathbf{k}}\, d$$

nd e efo e

$$\mathbf{k} \;-\; \int \sum_{\mathbf{l}\in\mathbb{Z}} |\;|\; e^{i\mathbf{k}}\, d$$

nd

$$\sum_{\mathbf{l}\in\mathbb{Z}} |\;| \;-$$

n e o n

$$\sum_{\mathbf{l}\in\mathbb{Z}} |$$

no d c o o ee e . . . . on e c ed e ; $\in \mathbf{Z}$ e e e $\{$ $_{j;k}$ = $^{j=}$ ^{j} − $\}_{k\ Z}$ fo n o ono of $\mathbf{W}_j$ e fo o n e D ec e , c c e ze ono e c po yno on of c co e pond o e o ono e of co p c y ppo ed e e n n o en

Lemma II.1 Any trigonometric polynomial solution of (2.26) is of the form

$$= \frac{1}{2} \quad e^{i} \quad e^{i}$$

where $M \geq$ **is the number of vanishing moments, and where is a polynomial, such that**

$$| \ e^{i} \ | = P \ \text{n} \ \tfrac{1}{2} \quad \text{n}^{M} \ \tfrac{1}{2} \quad \tfrac{1}{2}\,\text{co}$$

where

$$P\ y = \sum_{k}^{k\ M} \quad M - \quad ! \quad y^{k}$$

and is an odd polynomial, such that

$$\leq P\ y \quad y^{M} \quad \tfrac{1}{2} - \quad d \qquad f$$

e e $_{k}^{j}$ nd d_{k}^{j} y e e ed pe od c eq ence e pe od n j Co p n

nd ed y e py d e e e

$$
\begin{array}{ccccccc}
\{\ _{k}\} & \longrightarrow & \{\ _{k}\} & \longrightarrow & \{\ _{k}\} & \longrightarrow & \{\ _{k}\} \ \cdots \\
 & \searrow & & \searrow & & \searrow & \\
 & & \{d_{k}\} & & \{d & &
\end{array}
$$

e en de ne $f_{\mathbf{m}} := f_{\mathbf{m}} - \mathbf{m} \cdot f$ e e $\mathbf{m}$ c o en o $\langle f_{\mathbf{m}} \ \mathbf{M} \rangle :=$ fo
$:=$ M c e n e de ed o o on y o $\mathbf{M}$ y e con n e o

e e $\mathbf{V}_j^M$ en n e nd e p ce $\mathbf{W}_j^{M;}$ e o on co p e en of $\mathbf{V}_j^{M;}$ n $\mathbf{V}_j^{M;}$

$$\mathbf{V}_j^{M;} = \mathbf{V}_j^{M;} \quad M \quad \mathbf{W}_j^{M;}$$

e p ce $\mathbf{W}^{M;}$ p nned y e o ono

$$\{ \; i \quad l \, y \quad i \quad l \, y \quad i \quad l \, y \quad = \quad M\}$$

e e $\{\;_m\}_m^{m\;M}$ n o ono fo $\mathbf{V}^M$ c e e en on e M e e en n n o en e o on o e po yno ${}^i y^l$ = $M -$

e p ce $\mathbf{W}_j^{M;}$ p nned y d on nd n on of e f nc on of $\mathbf{W}^{M;}$ nd e of $\mathbf{L}$ con of e f nc on nd e o o de po yno ${}^i y^l$ = $M -$

e no e e o d en on e e eq e M d en co n on of one d en on f nc on e e M e n e of n n o en On e o e nd e o d en on e o ned y n co p c y ppo ed e e eq e on y ee c co n on c p e e con c on of e non nd d fo e ec on

II.5 A remark on computing in the wavelet bases

n y e no e once e e een c o en co p e e y de e ne e f nc on nd nd e efo e e e on n y n n e e n o e on n p ope y con c ed o e f nc on nd e ne e co p ed D e o e ec e de n on of e e e e n p on e pe fo ed e q d e o e nd e en f ey n o e q n e c ed nd A n e p e e co p e e o en of e n f nc on e e p e on fo e o en

$$\mathcal{M}^m = \int \;^m \; d \quad = \quad M -$$

n e of e e coe c en $\{\;_k\}_k^{k\;L}$ y e fo nd n fo fo

$$= \prod_j \quad {}^j$$

e e

$$= \sum_k^k \;_k e^{ik}$$

e o en $\mathcal{M}^{m}$ e o ned n e c y n e de ed cc cy y ec e y ene n eq ence of ec o $\{\mathcal{M}_{r}^{m}\}_{m}^{M}$ fo $r =$

$$\mathcal{M}_{r+}^{m} = \sum_{j}^{m} \quad ! \quad {}^{j r}\mathcal{M}_{r}^{m} \; {}^{j}\mathcal{M}^{j}$$

n

$$\mathcal{M}^{m} = \quad m \; \tfrac{1}{2} \sum_{k}^{k \; k} \quad k \quad m \quad = \quad M -$$

c ec o $\{\mathcal{M}_{r}^{m}\}_{m}^{M}$ ep e en M o en of e p od c n r e nd e on con e e p d y No ce e ne e co p ed e f nc on e f

e non nd d nd nd d fo

III.1 The Non-Standard Form

Le e n ope o

$$\mathbf{L} \quad \mathbf{R} \to \mathbf{L} \quad \mathbf{R}$$

e e ne y De n n p o ec on ope o on e p ce $\mathbf{V_j}$; $\in \mathbf{Z}$

$$P_{\mathbf{j}} \quad \mathbf{L} \quad \mathbf{R} \to \mathbf{V_j}$$

$$P_{\mathbf{j}} f \quad = \sum_{\mathbf{k}} \langle f \quad \mathbf{j;k} \rangle \quad \mathbf{j;k}$$

nd e p nd n n e e op c e e e o n

$$= \sum_{\mathbf{j} \quad \mathbf{Z}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j}$$ e

e e

$$\mathbf{j} = P_{\mathbf{j}} \quad - P_{\mathbf{j}}$$

e p o ec on ope o on e p ce $\mathbf{W_j}$ f e e e co e n en n e d of e e e

$$= \sum_{\mathbf{j}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j} \quad P_{\mathbf{n}} \quad P_{\mathbf{n}}$$

nd f e e ; = e ne e en

$$= \sum_{\mathbf{j}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j} \quad P_{\mathbf{n}} \quad P_{\mathbf{n}}$$ 7

e e $\sim$ $= P \quad P$ d e z on of e ope o on e ne e p n on e nd 7 deco po e ope o n o of con on f o d en e e

e non nd d fo ep e en on e 7 of e ope o c n of p e

$$= \{A_{\mathbf{j}} \quad B_{\mathbf{j}} \quad \mathbf{j}\}_{\mathbf{j} \quad \mathbf{Z}}$$

c n on e p ce $\mathbf{V_j}$ nd $\mathbf{W_j}$

$$A_{\mathbf{j}} \quad \mathbf{W_j} \to \mathbf{W_j}$$

$$B_{\mathbf{j}} \quad \mathbf{V_j} \to \mathbf{W_j}$$

$$\gamma_j : \mathbf{W_j} \to \mathbf{V_j}$$

e e e ope o $\{A_j, B_j, \gamma_j\}_{j \in \mathbb{Z}}$ e de ned $A_j := \ldots_j \ldots_j$, $B_j := \ldots_j P_j$ nd $\gamma_j := P_j \ldots_j$

e ope o $\{A_j, B_j, \gamma_j\}_{j \in \mathbb{Z}}$ d ec e de n on e e on

$$T_j := \begin{pmatrix} A_{j+} & B_{j+} \\ \gamma_{j+} & T_{j+} \end{pmatrix}$$

e e ope o $T_j := P_j T P_j$

$$T_j : \mathbf{V_j} \to \mathbf{V_j}$$

nd e ope o ep e en ed y e $\times$ n pp n

$$\begin{pmatrix} A_{j+} & B_{j+} \\ \gamma_{j+} & T_{j+} \end{pmatrix} : \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \to \mathbf{W_{j+}} \oplus \mathbf{V_{j+}}$$

f e e co e n en

$$:= \{\{A_j, B_j, \gamma_j\}_{j \in \mathbb{Z}, j \geq n}, T_n\}$$

e e $T_n := P_n T P_n$ f e n e of e n e en $:=$ n n nd e ope o e o nzed oc of e ee. e nd

Le e e fo o n o e on

- e ope o A_j de e e n e c on on e e on y nce e p ce $\mathbf{W_j}$ n n e e en of e d ec n

- e ope o B_j, γ_j n nd de e e n e c on e een e e nd co e ndeed e p ce $\mathbf{V_j}$ con n e p ce $\mathbf{V_{j'}}$ ee

- e ope o T_j n e ed e on of e ope o T_j

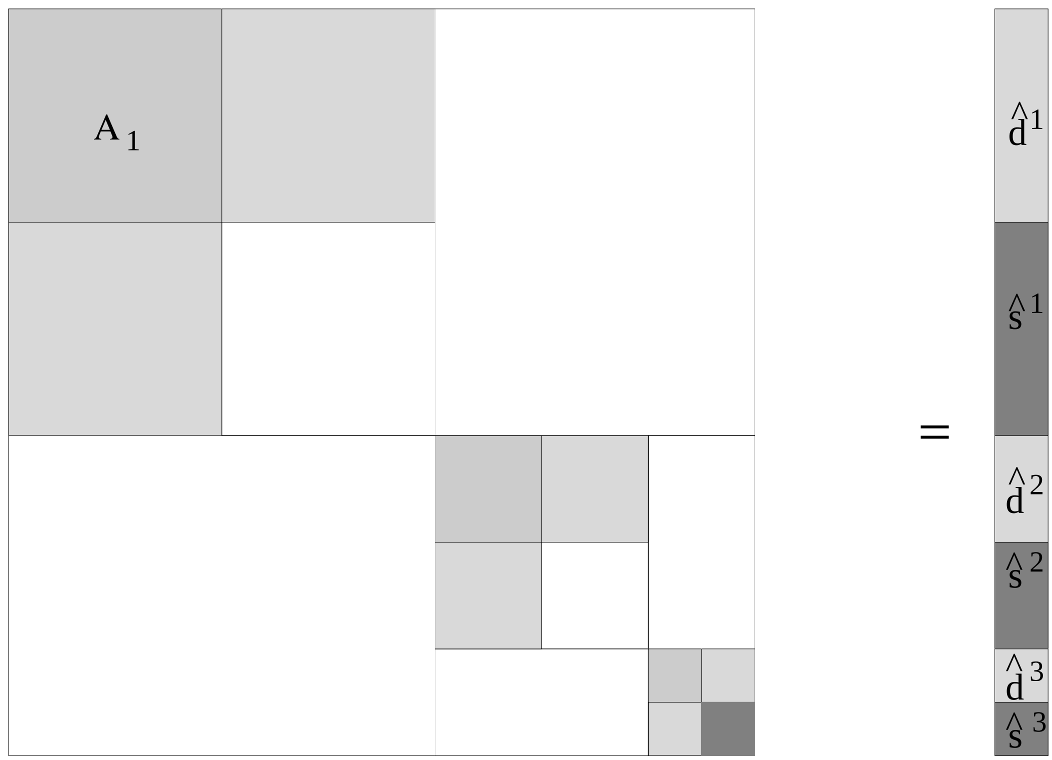
- e ope o A_j, B_j nd γ_j e ep e en ed y e ce α^j, β^j nd γ^j

$$\alpha^j_{k;k'} := \int\int \ldots (x, y) \ldots_{j;k} \ldots_{j;k'}(y)\, dx\, dy$$

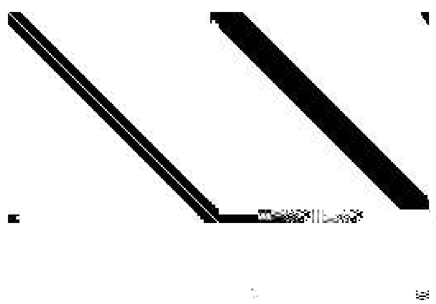
$$\beta^j_{k;k'} := \int\int \ldots (x, y) \ldots_{j;k} \ldots_{j;k'}(y)\, dx\, dy \qquad (7)$$

nd

$$\gamma^j_{k;k'} := \int\int \ldots (x, y) \ldots_{j;k} \ldots_{j;k'}(y)\, dx\, dy$$



A_1
=
$\hat{d}^1$
$\hat{s}^1$
$\hat{d}^2$
$\hat{s}^2$
$\hat{d}^3$
$\hat{s}^3$



e An e p e of n e non nd d fo ee p e

e ope o j ep e en ed y e j

$$ {}^{j}_{k;k'} = \int\int y \quad {}_{j;k} \quad {}_{j;k'} \; y \; d \; dy $$

en e of coe c en $_{k;k'}$ $' = N -$ epe ed pp c on of e [redacted]
fo 7 p od ce

[redacted]

$$ {}^{j}_{i;l} = \sum_{k;m}^{L} \quad k \; m \; {}^{j}_{k+ \; i+ \; ;m+ \; l+} $$

III.2 The Standard Form

e nd d fo o ned y ep e en n

$$\mathbf{V}_{\mathbf{j}} = \bigoplus_{\mathbf{j}'>\mathbf{j}} \mathbf{W}_{\mathbf{j}'}$$

nd con de n fo e c e e ope o $\{B_{\mathbf{j}}^{\mathbf{j}'}, \Gamma_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'>\mathbf{j}}$

$$B_{\mathbf{j}}^{\mathbf{j}'} : \mathbf{W}_{\mathbf{j}'} \to \mathbf{W}_{\mathbf{j}}$$

$$\Gamma_{\mathbf{j}}^{\mathbf{j}'} : \mathbf{W}_{\mathbf{j}} \to \mathbf{W}_{\mathbf{j}'} \quad 7$$

f e e e co e en n en n e d of e e

$$\mathbf{V}_{\mathbf{j}} = \mathbf{V}_{\mathbf{n}} \bigoplus_{\mathbf{j}'=\mathbf{j}+}^{\mathbf{n}} \mathbf{W}_{\mathbf{j}'}$$

n c e e ope o $\{B_{\mathbf{j}}^{\mathbf{j}'}, \Gamma_{\mathbf{j}}^{\mathbf{j}'}\}$ fo $\mathbf{j}' =$ n e e e n nd 7 nd n dd on fo e c e e e ope o $\{B_{\mathbf{j}}^{\mathbf{n}+}\}$ nd $\{\Gamma_{\mathbf{j}}^{\mathbf{n}+}\}$

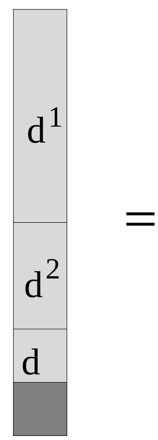
$$B_{\mathbf{j}}^{\mathbf{n}+} : \mathbf{V}_{\mathbf{n}} \to \mathbf{W}_{\mathbf{j}}$$

$$\Gamma_{\mathbf{j}}^{\mathbf{n}+} : \mathbf{W}_{\mathbf{j}} \to \mathbf{V}_{\mathbf{n}}$$

n no on $\Gamma_{\mathbf{n}}^{\mathbf{n}+} = \Gamma_{\mathbf{n}}$ nd $B_{\mathbf{n}}^{\mathbf{n}+} = B_{\mathbf{n}}$ f e n e of e e n e nd $\mathbf{V}$ n e d en on en e nd d fo ep e en on of $= P \quad P$

$$= \{A_{\mathbf{j}}, \{B_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'=\mathbf{j}+}^{\mathbf{n}}, \{\Gamma_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'=\mathbf{j}+}^{\mathbf{n}}, B_{\mathbf{j}}^{\mathbf{n}+}, \Gamma_{\mathbf{j}}^{\mathbf{n}+}\}_{\mathbf{j}\in\mathbf{n}}$$ re of o C f

e ope o e o n zed oc of e ee e nd e e f e ope o C de on Zy nd o p e do eratoj /R36 (3aV)Tj /R360 Td (Figu65


d1
d2
d
=

e co p e on of ope o

e co p e on of ope o o n o e o d e con c on of e p e ep e en on n o ono e d ec y ec e peed of co p on o e e co p e on of d of e fo e p e c e ed y e od o e n nd n p e ep e en on n e y e deq e fo e pp c on e co p e on of ope o c fo ep e en on n n o de o e ec e y co p e n e p e fo e nd d nd non nd d fo of ope o n e e e y e e ed co p e on e e fo de c of nd non nd d fo of ope the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| _{i;l}^{j}| \quad | _{i;l}^{j}| \quad | _{i;l}^{j}| \leq \frac{C_M}{| - |^{M+}}$$

for all $| - | \geq M$.

con de on pp y n e c e of p e do d en ope o Le e p e do d en ope o y o de ned y e fo

$$f \quad - e^{ix} \quad f \quad d - \quad y \ f \ y \ dy$$

e e e d on e ne of

Proposition IV.2 If the wavelet basis has M vanishing moments, then for any pseudo-di erential operator with symbol of and of satisfying the standard conditions

$$| _{x} | \leq C \; ; \quad | \quad +$$

$$| _{x} | \leq C \; ; \quad | \quad +$$

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| _{i;l}^{j}| \quad | _{i;l}^{j}| \quad | _{i;l}^{j}| \leq \frac{^{j} C_M}{| - |^{M+}}$$

for all integer , .

f e pp o e e ope o ^{N} y e ope o $^{N;B}$ o ned f o ^{N} y e n o ze o coe c en of ce $_{i;l}^{j}$ $_{i;l}^{j}$ nd $_{i;l}^{j}$ o de of nd of d $B \geq M$ o nd e d on en e y o e e

$$|| ^{N;B} - ^{N}|| \leq \frac{C}{B^{M}} \text{ o } N$$

e e C con n de e ned y e e ne n o n e c pp c on e cc cy of c c on ed nd e p e e of e o n o c e nd d B nd o de M e o e c o en n c nne e de ed p ec on of c c on c e ed f M ed en B o e c

$$|| ^{N;B} - ^{N}|| \leq \frac{C}{B^{M}} \text{ o } N \leq$$

y — y (4.2?)

e e f c y n y n e f nc on e e nd e nd e efo e e ope o L nd L po e o dec de f C de on Zy nd ope o o nded

Theorem IV.1 (G. David, J.L. Journe) **Suppose that the operator (3.1) satis es the conditions (4.5), (4.6), and (4.16). Then a necessary and su cient condition for to be bounded on L is that in (4.24) and y in (4.25) belong to dyadic BMO, i.e. satisfy condition**

$$\sup_J \frac{1}{|J|} \int_J | \quad - \quad_J | \, d \leq C \qquad (4.2?)$$

where is a dyadic interval and

$$\quad_J = \frac{1}{|J|} \int_J \quad d \qquad (4.27)$$

p n e ope o n o e of ee e e nd e n e ep e y e d o e e e e e no e e f nc on nd e e y co p ed n e p oce of con c n e non nd d fo nd nd y e ed o p o de f e e of e no of e ope o e d e en ope o n e e e

V.1 The operator d=dx in wavelet bases

e non nd d fo of e e co on y ed ope o y e co p ed e p c y e con c e non nd d fo of e ope o d d e e e en of A_j B_j nd r of e e e d d d

e e n e e oco e on coe c en of e e ={ k}k k L

$$n = \sum_{i}^{L} \quad i \quad i+n \qquad n = \qquad L -$$

e y o ee e oco e on coe c en n e en nd ce e ze o

$$k = \qquad = \qquad L \quad -$$

y e e fyed y n o co p e | nd |

$$| = - \sum_{n}^{L} \; n \cos n$$

$$| = - - - \sum_{k}^{L=} k \cos - - \sum_{k}^{L=} k \cos$$

e e n e en n Co n n nd o fy e o n

$$\sum_{k}^{L=} k \cos =$$

nd ence nd

e e en o en of e coe c en k f o n n e y

$$\sum_{k}^{k \; L=} k \quad - \quad m = \quad \text{fo} \quad \leq \quad \leq M -$$

e

nce

$$- \quad m \quad | \quad = \quad \text{fo} \quad \leq \quad \leq \; M -$$

c fo o f o e Co po d en f e 7 e d e y fy o 7 7 e f d cof e n $:=$ e e e 7

$$r_i := \sum_{k}^{L} \sum_{m \, k}^{k+} \, k \; k \; m \, r_{i+m}$$

C n n e o de of on n nd n e f c $\sum_{k}^{L} k :=$ e e

$$r_l := r_l \sum_{n}^{L} n \; r_{l \, n} \; r_{l+n} \quad \in \mathbb{Z}$$

e e n e en n n e o n e f o n o de o o n e e fo o n e on

$$\sum_{l}^{l} m - := m \sum_{l}^{l \, m} - l \quad M_l' \; m \; l$$

e e

$$M_l' := \int^{\mathbb{Z}} + l \, d :=$$

e e o en of e f nc on e on fo o p y on n o e n fo nd n Le n z e n nd $:=$ e o n f $M \geq$ en

$$| \; | \; | \leq C \quad |$$

e e nd ence e n e n o e y con e en on fo o f o Le of e e o n

$$| \; | \leq C \quad |^{M+\log_2 B}$$

e e

$$B := \sup_{\mathbb{R}} | \; e^i \; |$$

D e o e cond on e e o $B := M - \; -$ o e e e ence of o on of e y e of eq on e nd fo o f o e e ence of e n e n nce e c n f nc on co p c ppo e e e on y 7 7 d ee e e 7 d o e e e

$$r(\xi) = \sum_l r_l e^{il\xi}$$

$$r_{\mathrm{even}}(\xi) = \sum_l r_{l} e^{il\xi}$$

7

nd

$$r_{\mathrm{odd}}(\xi) = \sum_l r_{l+} e^{i(l+)\xi}$$

No c n

$$r_{\mathrm{even}}(\xi) = r(\xi) \quad r(\xi)$$

nd

$$r_{\mathrm{odd}}(\xi) = r(\xi) - r(\xi)$$

nd n e o n f o

$$r(\xi) = r(\xi) \quad r(\xi) \quad | - r(\xi) - r(\xi)$$

n y n e e

$$r(\xi) = | r(\xi) | r(\xi)$$

e n = n e e o n r = r nd e n q ene of e on of e nd fo o f o e n q ene of e ep e en on of d d en e on r_l of e nd e con de e ope o $_j$ de ned y e coe c en on e p ce $\mathbf{V}_j$ nd pp y o c en y oo f nc on f nce $r^j_l = {}^j r_l$ e e e

$$_j f = \sum_{k \in Z} {}^j \sum_l r_l f_{j;k\, l} \quad {}_{j;k}$$

e e

$$f_{j;k\, l} = {}^{j=} \int f \quad {}^j - \quad d$$

e n ee

$$f_{j;k\, l} =$$

d 7 7

e e $= - j$ nd $| - | \le j$ n e n e nd n e
nd e o n

$$j f = \sum_{k \in Z} \int_{+} f' \; j;k \; d \; j;k$$

$$j \sum_{k \in Z} \sum_{l} - r_l \int_{+} f'' \; j;k \; d \; j;k \; ! \qquad 7$$

c e $\to -\infty$ ope o j nd $d\,d$ co nc de on oo f nc on n
nd d en e y o p o e $= d\,d$ nd ence e o on o e
nd n q e e e on 7 fo o no f o

Remark 2 e no e e p e on nd fo l nd $l \; l = - l$ y
e p ed y c n n e o de of on n nd nd n od c n
e co e on coe c en $\sum_{i}^{L} n_i \; i+n$ $\sum_{i}^{L} n_i \; i+n$ nd $\sum_{i}^{L} n_i \; i+n$ e
e p e on fo l e pec y p e $l = e r \; l - r_l$

Examples. o e e p e e D ec e e e con c ed n
e co p e e coe c en k $=$ M e e M e n e of n n
o en nd $L = M$ n e on e of

$$| = - \frac{M -}{M - \;\; M} \int n^{M} d \qquad e$$

e nd y co p n ^{R} n ^{M} d

$$| = - C_M \sum_{m}^{M} \frac{- \; m \; co \; -}{M \quad M \; - \quad -} \qquad e$$

e e

$$C_M = \left[\frac{M -}{M - \; e^{M}} \right]$$

y co p n e nd e e

$$m = \frac{- \; m \; C_M}{M \quad M \; - \quad -} \qquad e e = M$$

e no e y e of e n o on of ne y e on co
e c en m n e on y con c on e coe c en r_l e on
n e e coe c en r_l e e e fo D ec e e e ed n e
of n n o en M e e e e e e e e fo en

j e d e d en

j ene e
of ep d o yno e

o n eq on of opo on e p e en e e fo D ec e e e
M e

1 M

nd

r r

e coe c en of e p e c n e fo nd n ny oo
on n e c n y c o ce of coe c en fo n e c d en on

---

2 M

nd

r r r r_4

---

3 M e

nd

r r r

r_4 r r

---

4 M

nd

r r r

r_4 r

5 M =

and

r_4 =

Coe c en fo M = nd M = c n e co p ed e co e pond n o p f o e fo o n e e o

Iterative algorithm for computing the coefficients r_l.

A y of o n eq on e nd e y o e n e e o e r = nd r = nd e e n e o eco p e r_l e y o e fy n e nd e ed d e o e c o ce of n z on e fo o n fo D ec e e e M = co p ed n o d p y e coe c en $\{r_l\}_l$ e no e r_l = $-r_l$ nd r =

V.2 The operators d^n/dx^n in the wavelet bases

o e ope o $d\ d$ e non nd d fo of e ope o $d^n\ d^n$ co p e e y de e ned y ep e en on on e p ce **V** e y e coe c en

$$r_l^{(n)} = \int \; \frac{d^n}{d^n} \; d \quad \in \mathbf{Z}$$

o e n e y

$$r_l^{(n)} = \int \; ^n | \; | \; e^{il} \; d$$

f e ne n o e ee e e o

	i	Coe cients
M = 5	1	-0.82590601185015
	2	0.22882018706694
	3	-5.3352571932672E-

	i	Coe cients
M = 8	1	-0.88344604609097
	2	0.30325935147672

Proposition V.2 **1. If the integrals in (5.52) or (5.53) exist, then the coe cients $r_l^{n)}$, $\in$ Z satisfy the following system of linear algebraic equations**

$$r_l^{n)} \;=\; {}^{n} \left[r_{\,l} \;-\; \sum_{k}^{L=} \;{}_{k}\; r^{\,n)}_{\,l\;\;k+} \;\; r^{\,n)}_{\,l+\;k} \right] \qquad\qquad (\text{?})$$

and

$$\sum_{l} {}^{n} r_l^{n)} \;=\; - \;\; {}^{n} n$$

where $_{k}$ are given in (5.19).

2. Let $M \geq \; n$, where M is the number of vanishing moments in (2.16). If the integrals in (5.52) or (5.53) exist, then the equations (5.54) and (5.55) have a unique solution with a nite number of non-zero coe cients $r_l^{n)}$, namely, $r_l^{n)} \neq$ for $-L \quad \leq \; \leq L -$. Also, for even n

$$r_l^{n)} = -r^{\,n)}_{\,l}$$

$$\sum_{l} {}^{n} r_l^{n)} = \quad n = \qquad n \quad - \qquad\qquad (7)$$

and

$$\sum_{l} r_l^{n)} =$$

and for odd n

$$r_l^{n)} = -r^{\,n)}_{\,l}$$

$$\sum_{l} {}^{n} \; r_l^{n)} = \quad n = \qquad\qquad \leq \; \leq L$$

$$A \quad e \; M$$

e no e on e e e $L =$ e e e e o n n o en $M =$ do no e e e o de e ponen ee , e ep e en on of e d de e e on y f e n e of n n o en $M =$

e eq on fo co p n e coe c en $r_l^{n)}$ y e e ed n e en e p o e Le de e e eq on co e pond n o e fo d^n d^n d ec y f o e e e

$$r_l^{n)} = \int \sum_{k \in Z} | \quad |^n \quad {}^n e^{il} \, d$$

e efo e

$$r = \sum_{k \in Z} | \quad |^n \quad {}^n$$

e e

$$r = \sum_l r_l^{n)} e^{il}$$

n e e on

e

n o e nd de of nd n o e e en nd odd nd ce n ep e y e e

$$r = {}^n \quad | \, r \quad | \, r$$

Le con de e ope o M on pe od c f nc on d f n f d

$$M f = \quad | \, f \quad | \quad |$$

n e e e e n e c e dence n f c of n e e nce ep
e en one of e d n e of co p n n e e e e
e 7 e c e

N	[illegible]	[illegible]$_p$
64	0.14545E+04	0.10792E+02
128	0.58181E+04	0.11511E+02
256	0.23272E+05	0.12091E+02
512	0.93089E+05	

e con o on ope o n e e e

n ec on e con de e co p on of e non nd d fo of con o on ope o . o con o on ope e q d e fo fo ep e en n e e ne on **V** e of e p e fo d e

n nd e den y fo o f o
o d

nce e o en of e f nc on n eq on e e d o
po n q d e fo fo co p n e ep e en on of con o on
e ne e fo co p c y ppo ed e e fo o
e e nne fo e pec c o ce of e e e de c eqn
e e e f ed o en of e f nc on n e ef p pe fo
e de

e e e n od ce d en pp o c c con n of
ne e c eq on ec o y p o c cond on pe
c y p e f e y o of e ope o o o eneo of p e
c e e ope o co p e e y de ned y ep e en on on **V**

Le con de o e p e of c ope o e nd
ope o of f c on d en on o n d en on

VI.1 The Hilbert Transform

e pp y o e od o e co p on of e non nd d fo e n
fo

$$\mathcal{H}f \quad y \quad p \int \frac{f}{} d$$

e e p deno e p nc p e

e ep e en on of $\mathcal{H}$ on **V** de ned y e coe c en

$$r_{\mathbf{l}} \int \mathcal{H} \quad d \quad \in \mathbf{Z}$$

c n n co p e e y de ne o e coe c en of e non nd d
$\mathcal{H} \{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j} \; \mathbf{Z}}$ $A_{\mathbf{j}} \; A$ $B_{\mathbf{j}} \; B$ nd $\mathbf{j}$ e e e

	l	Coe cients r_l	l	Coe cients r_l
M = 6	1	-0.588303698	9	-0.035367761
	2	-0.077576414	10	-0.031830988
	3	-0.128743695	11	-0.028937262
	4	-0.075063628	12	-0.026525823
	5	-0.064168018	13	-0.024485376
	6	-0.053041366	14	-0.022736420
	7	-0.045470650	15	-0.021220659
	8	-0.039788641	16	-0.019894368

e e e coe c en r_l of e n fo fo D ec e e e n n o en

e coe c en $r_l \in$ **Z** n fy e fo o n y e of ne e c eq on

$$r_l = r_l - \sum_{k}^{L} k \; r_{l-k} + r_{l+k}$$

e e e coe c en k e en n n e nd 7 e o n e y p o c of r_l fo e

$$r_l = O \frac{}{M}$$

By e n n e of

$$r_l = \int | | n \, d$$

e o n $r_l = r_l$ nd r e no e e coe c en r c nno e de e ned f o eq on e nd

o n e e y p o c cond on e co p e e coe c en $r_l \neq$ ny p e ed cc cy

Example.

e co p e e coe c en r_l of e n fo fo D ec e e e n n o en cc cy e coe c en fo e o ned n y p o c e no e $r_l = r_l$ nd r

VI.2 The fractional derivatives

e e fo o n de n on of f c on de e

$$ {}_{x}f \;=\; \int_{+} \frac{-\,y\;+}{-} f\; y\; dy $$

e e e con de $\neq$ f en de ne f c on n de e
e ep e en on of $_{x}$ on $\mathbf{V}$ de e ned y e coe c en

$$ r_{l} = \int_{+} - \quad {}_{x} \quad d \quad \in \mathbf{Z} $$

p o ded n e e
e non nd d fo $_{x} = \{A_{j}\; B_{j}\; \;_{j}\}_{j \; Z}$ co p ed $A_{j} = {}^{j}A \quad B_{j} = {}^{j}B$ nd $_{j} = {}^{j}$ e e e e en $_{i\;l}$ $_{i\;l}$ nd $_{i\;l}$ of $A \quad B$ nd e o ned f o e coe c en r_{l}

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} {}_{k}\;{}_{k'}\; r_{i+k\;k'} $$

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} {}_{k}\;{}_{k'}\; r_{i+k\;k'} $$

nd

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} {}_{k}\;{}_{k'}\; r_{i+k\;k'} $$

e y o e fy e coe c en r_{l} fy e fo o n y e of ne e c eq on

$$ r_{l} = \left[r_{l} - \sum_{k}^{L} {}_{k} \quad r_{l\;k+} \quad r_{l+\;k} \right] $$

e e e coe c en $_{k}$ e en n n e nd e o n e y p o c of r_{l} fo e

$$ r_{l} = \frac{}{-}\;\frac{}{+} \quad O\;\frac{}{+\;+\;M} \quad \text{fo} $$

$$ r_{l} = \quad \text{fo} \quad e $$

Example.

		Coe cients		Coe cients
	i		i	
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	-1.91718816E-02
	-4	-4.34633415E-03	7	-1.52272841E-02
	-3	2.28821728E-02	8	-1.24667403E-02
	-2	-8.49883759E-02	9	-1.04479500E-02
	-1	0.27799963	10	-8.92061945E-03
	0	0.84681966	11	-7.73225246E-03
	1	-0.69847577	12	-6.78614593E-03
	2	2.36400139E-02	13	-6.01838599E-03
	3	-8.97463780E-02	14	-5.38521459E-03

M p c on of ope o n e e e

VII.1 Multiplication of matrices in the standard form

e p c on of ce of C de on Zy nd nd p e do d e en ope o n e nd d fo eq e o O N o N ope on n dd on po e o con o e d of e n e nd y e n o ze o e en e n e p od c e o e o d of nd e efo e

$$|| \cdot - \cdot || \leq$$ 7

e nd de of 7 do n ed y , o e p e f e co p e [4] en e o e one n c n d

VII.2 Multiplication of matrices in the non-standard form

e no o ne n o fo e p c on of e ope o n e non nd d fo ne o e e n e y deco pe e e n e p oce of p c on Le nd e o ope o

$$\cdot \quad \mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{L} \quad \mathbf{R}$$ 77

en e non nd d fo of nd $\{A_j \; B_j \; \cdot_j\}_{j \; z}$ nd $\{A_j \; B_j \; \cdot_j\}_{j \; z}$ e co p e e non nd d fo $\{A_j \; B_j \; \cdot_j\}_{j \; z}$ of $\cdot - \cdot$

e ec e ope o of

n y e e e 7 of o e

7

e e

$$\sum_{j}^{n} A_j A_j \quad B_j \quad B_j \quad j \quad A_j B_j \quad j \quad A_j \qquad (7)$$

nd

$$n \quad n \quad \sum_{j}^{n} P_j \, B_j P_j \qquad (7)$$

e ope o n e 7 e c n on fo o n e p ce

$$A_j A_j \quad B_j \quad \mathbf{W}_j \to \mathbf{W}_j \qquad (7)$$

$$B_j \quad j \quad A_j B_j \quad \mathbf{V}_j \to \mathbf{W}_j \qquad (7)$$

$$j \quad A_j \quad \mathbf{W}_j \to \mathbf{V}_j \qquad (7)$$

nd e ope o n e 7

$$B_j \quad \mathbf{V}_j \to \mathbf{V}_j \qquad (7\ 7)$$

e e n 7 d d e f d 7 7 f

of ope on ed e c e e o o e p e c e e o n e of
ope on ep p opo on o N
e e n ope o A_j B_j j n tot49 -410.315ac5 0 Td (36su8712.7097 0552

e e o n e e e

e f p c on o of ec on e econd fe o n e of e e o

VIII.1 An iterative algorithm for computing the generalized inverse

n o de o

p oced e nd e e on n e e e e n fo e co p on e e pe fo ed on n p c o on nd e ed o ne f o L N AC fo co p n e n e deco po on o e e ed e fo o n f n

$$A_{ij} = [illegible]$$

e e = N e cc cy e o d o [4] e en e of X_k e o [4] e e y e c y e o ed f e e c e on

Size $N \times N$	SVD	FWT Generalized Inverse	L_2-Error
128 × 128	20.27 sec.	25.89 sec.	3 1 · 10^{-4}
256 × 256	144.43 sec.	77.98 sec.	3 42 · 10^{-4}
512 × 512	1,155 sec. (est.)	242.84 sec.	6 0 · 10^{-4}
1024 × 1024	9,244 sec. (est.)	657.09 sec.	7 7 · 10^{-4}
...	...	...	...
$2^{15} \times 2^{15}$	9.6 years (est.)	1 day (est.)	

Le de e e e e e o nd c n n e c f nc on c c ope o c n e p e en ed e c en y e fo p e do d en op e o N e c e nd e e pe fo nce of e o e epo ed ep e y

VIII.2 An iterative algorithm for computing the projection operator on the null space.

Le con de e fo o n e on

$$X_{k+1} = X_k - X_k [illegible]$$

$$X = A A [illegible]$$

e e A e d o n nd c o en

en $-X_k$ con e e o P_{null} c n e o n e e d ec y o y co n n n n n ep e en on fo $P_{null} := -A\ AA^{\dagger}A$ e e on o co p e e ene zed n e $AA^{\dagger}$ e f p c on o e e e on e f fo de c of ope o e e co p e y e o fo e ene zed n e e po n d ence o e e e doe no eq e co p e y of e n e ope o on y of e po e of e ope o

VIII.3 An iterative algorithm for computing a square root of an operator.

Le de e n e on o con c o $A^{=}$ nd $A^{=}$ e e A fo p c y e f d o n nd non ne e de n e ope o e con de e fo o n e on

$$Y_{l+} := Y_l - Y_l X_l Y_l$$

$$X_{l+} := -X_l\ Y_l A$$

$$Y := -A$$

$$X := -PA$$

7

e e c o en o e e e en e of $-A$ e n $\sqrt{\ }$ e eq ence X_l con e e o $A^{=}$ nd Y_l o $A^{=}$ By n $A := V\ DV$ e e D d on nd V n y e y o e fy o X_l nd Y_l c n e en $X_l := V\ P_l V$ nd $Y_l := V\ _l V$ e e P_l nd $_l$ e d on nd

$$_{l+} := \ _l - \ _l P_l \quad _l$$

P e e

r n f f f d e y A f d f d

VIII.4 Fast algorithms for computing the exponential, sine and cosine of a matrix

e e ponen of o n ope o e ne nd co ne f nc on e on e o e con de ed n ny c c of ope o A n e c e of e ene zed n e

X Co p n $F(u)$ n e e e e

n ec on e de e f d p e o fo co p n e e n n n e y d en e f nc on nd ep e en ed n e e An po n e p e O n y c e ene ze eo e of M Bony on e p op on of n e of o on of non ne eq on O n e c pp o c o e e no e e e pec de n e of pp c on of o

IX.1 The algorithm for evaluating u^2

e n o o co p e Le p o ec $\in$ **L R** on p ce **V**$_j$ $\in$ **Z** o

$$_j \,-P_j \qquad _j \in \mathbf{V}_j$$

n o de o deco p e e e e e e e op c e e

$$- \;_n - \sum_j^n \Big[P_j \quad - P_j \Big] - \sum_j^n P_j \quad P_j \quad P_j \quad - P_j$$

n P_j $-P_j$ $_j$ e o n

$$- \;_n - \sum_j^n P_j \quad _j \quad _j$$

o

$$- \sum_j^n P_j \quad _j \qquad \sum_j^n \quad _j \quad _j \qquad _n$$ e

n e e e no n e c on e een d en e nd

o e n e c p po e need fo o e n e n e of e o c e

Befo e p oceed n f e e con de n e p e of e n [redacted]
e e e fo o n e p c e on

$$j \quad k \quad = \quad j= \quad j \quad k$$
$$j \quad k \quad = \quad j= \quad j \quad k \qquad 7$$
$$j \quad k \quad j \quad k \quad = \quad j= \quad j \quad k$$

A o e pod c on e e e e ze o
p nd n e p c y n o [redacted]

$$= \sum_{j}^{n} \sum_{k \in Z} d_k^j \, {}^j_k \qquad \sum_{k \in Z} {}^n_k \, {}^n_k$$

nd n 7 e o n f o e

$$= \sum_{j}^{n} j= \sum_{k \in Z} d_k^j \, {}^j_k \, {}^j_k \qquad \sum_{j}^{n} j= \sum_{k \in Z} d_k^j \, {}^j_k \qquad n= \sum_{k \in Z} {}^n_k \, {}^n_k$$

On deno n

$$d_k^j = j= + d_k^j \, {}^j_k$$
$${}^j_k = j= d_k^j$$
$${}^n_k = n= {}^n_k$$

e e e

$$= \sum_{j}^{n} \sum_{k \in Z} d_k^j \, {}^j_k \qquad \sum_{j}^{n} \sum_{k \in Z} {}^j_k \, {}^j_k \qquad \sum_{k \in Z} {}^n_k \, {}^n_k$$

e no e **if the coe cient d_k^j is zero then there is no need to keep the corresponding average** ${}^j_k$ n o e o d e need o eep e e on y ne e n e e e e e e e coe c en d_k^j o p od c ${}^j_k d_k^j$ e n c n fo en cc cy

of coe c en c need o e o ed y e ed ced f e y o e n fo e p e

$$M^{j;j'}_{WWW} \quad ' \quad - \quad j'= \int + \quad j \; j' \quad {}^{j}_{k} \, {}^{j'}_{k'} \quad {}_{j-j'} k \; l \quad d$$

o

$$M^{j;j'}_{WWW} \quad ' \quad - \quad j'= M^{j \; j'}_{WWW} \quad - \quad ' \quad j \; j' \quad -$$

o e e e o n c n ed c on n e n e of coe c en con e q ence of e f c e coe c en n dec y e d nce r $-$ $-$ $'$ e een e e f e n e of n c n coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ p opo on o e n e of c e o N o e e n e of ope on eq ed o e e e pp n

o c e nece y o o e on y o e e $_{\mathbf{k}'}^{\mathbf{j}}$ c co ne e n c n coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ o p od ce non ze o con on e efo e c en o o e on y o $_{\mathbf{k}'}^{\mathbf{j}}$ fo c e e e e coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ c $| - '| \leq$ nd e p od c $_{\mathbf{k}'}^{\mathbf{j}} d_{\mathbf{k}}^{\mathbf{j}}$ o e e e o d of cc cy e n e need o o e e e on y n e ne o ood of n e

e n e of ope on fo e p nd n of e econd e n n o e e e p opo on o e n e of n c n en e nd e e e co p e e y o fo

Remark. e o fo e on n e e e o o e e e p od c of o f nc on nce $\frac{}{4}$ — —

IX.2 The algorithm for evaluating $\mathbf{F}(\mathbf{u})$

Le e n n n e y d en e f nc on n o de o deco p e e e e n e e op c e e

$$- \quad {}_{\mathbf{n}} \quad \sum_{\mathbf{j}}^{\mathbf{j}\;\mathbf{n}} \quad P_{\mathbf{j}} \quad - \quad P_{\mathbf{j}} \qquad 7$$

p nd n e f nc on n e y o e e e po n y e P d 7 7 e de p

e no e e e no e e econd de e of n e e e no e en
de e n n nd con de n e e nde of e e e n
n e o e e o n e e of M Bony e e o e o e e e
y oo e n n Bony e nce e e nde f c o n e d
of e y eep o e e o e e e nde y oo
no c e po n f e e n d n e n co p n
o epe ed pp c on of e o fo c en o co p e o
f nc on o e e e e e e e n y c d n e n con de n n
p c y n eo e c c e z n e

efe ence

. , B A pe p e ep e en on of oo ne ope o D e Y e n e y

. , C e L een d nd o n A f d p e po e o fo p c e on **SIAM Journal of Scienti c and Statistical Computing** e Y e n e y ec n c epo YALeB DO o e

e A Co en D ec e nd C .

M e e of f eq ency c nne deco po on of e nd e e ode ec n c epo e Co n n e of M e c c ence Ne Yo n e y

Y Meye Le c c en q e e onde e e e e o en q d e C MAD n e e D p ne

Y Meye nc pe d nce de e e enne e e e d ope e n