Department of pplied Mathematics
reliminary Examination in Numerical nalysis
ugust, 2013

January 29, 2014

Solutions:

1. **Root Finding.**
   We ant to find a function such that the e ton iterations $x_{n+1} = x_n \quad f(x_n)/f^0(x_n)$ 'hop about' forever ithin a finite interval, ithout ever converging. The easiest example ould seem to be if the iterates form some short c cle, the simplest of all such arising if $x_{n+1} = \quad x_n$, i.e. $\quad x_n = x_n \quad f(x_n)/f^0(x_n)$. Simplif ing the notation b riting x in place of x_n, this ill be satisfied if $f^0(x) = \frac{1}{2x} f(x)$. We can thus choose $f(x) = \begin{cases} c\sqrt{x} & \text{if } x \quad 0 \\ c\sqrt{\quad x} & \text{if } x < 0 \end{cases}$, here c is an arbitrar constant.

2. **Numerical uadrature.**

   (a) Let h denote the length of a single subinterval before the extrapolation is done. Including also the subinterval midpoint, the trapezoidal rule over this subinterval ould have the eights at its ends and midpoint: $T_0 = h[\frac{1}{2}\ 0\ \frac{1}{2}]$ and, hen using also the midpoint $T_1 = h[\frac{1}{4}\ \frac{1}{2}\ \frac{1}{}$

3. **Interpolation/Approximation.**

We start by multiplying the numerator and denominator of $p_n(x)$ by $\ell_n(x)$, to obtain

$$p_n(x) = \frac{\sum_{j=0}^{n} w_j f(x_j)(x - x_0)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{\sum_{j=0}^{n} w_j (x - x_0)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}.$$

Note next that $\ell'(x)$ will become a sum of $n+1$ terms, all but one vanishing when substituting $x \to x_j$. Hence,

$$\ell'(x_j) = (x_j - x_0)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n).$$

Substituting $w_j = 1/\ell'(x_j)$ into the expression for $p_n(x)$ above thus gives

$$p_n(x) = \frac{\sum_{j=0}^{n} f(x_j) \; (x - x_0 \ldots}{}$$

0.90914 -246.701 -3.0293 -304 0.71516 -2.970 -15 -3.62 -23.19 -13.246 -1.97 -2.152 -23.1 [:)T

5. **ODEs**

(a) We have

$$\mathbf{f}(t_n + h, \mathbf{y}_n + \mathbf{k}_1) = \mathbf{f}(t_n, \mathbf{y}_n) + h \left(\frac{\partial \mathbf{f}(t_n, \mathbf{y}_n)}{\partial t} + \frac{\partial \mathbf{f}(t_n, \mathbf{y}_n)}{\partial \mathbf{y}} \mathbf{f}(t_n, \mathbf{y}_n) \right) + O(h^2), y_n + \mathbf{k}$$

. **PDEs**

We look for the solution in the form

$$u(x;y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} u_{mn} \sin(\ (m+1)\,x) \sin\ \ n + \frac{1}{2}\ \ y$$

so that

$$\frac{\partial u}{\partial y}(x;y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} u_{mn}\ \ n + \frac{1}{2}\ \ \sin(\ (m+1)\,x) \cos\ \ n + \frac{1}{2}\ \ y$$

satisfies the eumann boundar at $y = 1$. Computing

$$u(x;y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} u_{mn}\ ^2\ (m+1)^2 + \ n + \frac{1}{2}\ ^2 ! \sin(\ (m+1)\,x) \sin\ \ n + \frac{1}{2}\ \ y\ \ ;$$

e seek an expansion of the right hand side,

$$f(x;y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_{mn} \sin(\ (m+1)\,x) \sin\ \ n + \frac{1}{2}\ \ y$$

so that e can set

$$u_{mn} = \frac{f_{mn}}{^2\ (m+1)^2 + \ n + \frac{1}{2}\ ^2}:$$

Consider $x_k = (k+1)=M$, $k = 0;:::M\ \ 1$ and $y_l = (l+1)=N$, $l = 0;:::N\ \ 1$ so that

$$f(x_k;y_l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_{mn} \sin\ \frac{(m+1)\,k}{}$$

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