Solutions

Problem 1:

(a) What does it mean for an operator to be compact? A linear operator $T : H \to H$ is compact if $T(B)$ is a precompact subset of $H$ for every bounded subset $B \subset H$ (recall "precompact" means its closure is compact, or equivalently, that every sequence has a convergent subsequence). That is to say for every bounded sequence $(x_n) \subset H$, then $(Tx_n)$ has a convergent subsequence.

(b) Discuss convergence: Note that the problem doesn't ask the student to prove if the limit is in $B(H)$, so this may be assumed.

(a) We show convergence in norm is sufficient.

Solution 1 Let $(x_m) \subset H$ be a bounded sequence with $\|x_m\| \leq B$ for all $m$. We will show that there is a subsequence $(m_k)$ such that $(Ax_{m_k})$ is Cauchy, and since $H$ is complete, therefore it is convergent. The only tricky part is defining $m_k$. Since $A_1$ is compact, there is a subsequence $m_{k(1)}$ such that $A_1(x_{m_{k(1)}})$ is convergent (to, say, $y_1$). Since $A_2$ is compact, there is a subsequence $m_{k(2)}$ of $(m_{k(1)})$ such that $A_2(x_{m_{k(2)}})$ is convergent to $y_2$ (and $A_1(x_{m_{k(2)}})$ is still convergent to $y_1$, since this is a subsequence of the subsequence).

For each $k$, we have a subsequence of the subsequence associated with $k-1$. We can take the $k^{th}$ term of this new subsequence, and make this into a master subsequence $(m_k)$. This is known as the diagonalization trick. Since this master subsequence is bounded, and $\|A_n - A\| \to 0$, an $\epsilon/3$ argument shows that the sequence $(y_k)$ is Cauchy, and thus there is some $y$ with $y_k \to y$, and then again using an $\epsilon/3$ argument we see that $Ax_{m_k} \to y$, thus proving that $A$ is a compact operator.

Solution 2 A slicker proof is using the fact that a compact operator can be arbitrarily well-approximated by a finite-rank operator; using this, the proof is trivial (basically, that's what this problem is trying to show).

Solution 3 Use the fact that a compact operator (on a Hilbert space) maps weakly convergent sequences to strongly convergent ones, i.e. if $A_n$ is compact, then $x_k \rightharpoonup x$ implies $A_n x_k \to A_n x$. Thus we only need to show $Ax_k \to Ax$. We do this with the usual triangle inequalities:

$$\|Ax_k - Ax\| \leq \|Ax_k - A_n x_k\| + \|A_n x_k - A_n x\| + \|A_n x - Ax\|$$

and we can make all terms small. But note that we require norm convergence and boundedness in order for the first and third terms to be BOTH small. If we have only strong convergence, then we can make them small separately (by choosing $n$ large enough) but not necessarily have both of them small. The middle term is arbitrarily small by choosing $k$ sufficiently large.

Solution 4 Let $B \subset H$ be bounded, so for every $n$, $A_n(B)$ is pre-compact and hence totally bounded. It is sufficient TJ/F33 10.9091 Tf 31.667 0 Td [( )-278(>)]TJ/F15 10.9091

For any $x \in B$, we have

$$\|(A_n - A)x\| < \epsilon/(3M)\|x\| \le \epsilon/3.$$

Hence if we pick an arbitrary point $A(x) \in A(B)$, it is within $\epsilon/3$ of the point $A_n(x) \in A_n(B)$. By the triangle inequality, since $(x_i)$ is an $\epsilon/3$ net for $A_n(B)$, there is some $Ax_i$ that is within $\epsilon$ of $A(x)$.

Explicitly, for $x \in B$, there is some $x_i$ such that

$$\begin{aligned}\|Ax - Ax_i\| &\le \|Ax - A_nx\| + \|A_nx - A_nx_i\| + \|A_nx_i - Ax_i\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

Hence $\{Ax_i\}$ is a finite $\epsilon$-net for $A(B)$, and since $\epsilon$ was arbitrary, this means $A(B)$ is totally bounded, hence pre-compact.

(b) We show strong convergence is not sufficient. Take $A_n$ to be defined as in Example 5.46 in the book, where for $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$

Now, to evaluate the limit of the integrand, use standard techniques (e.g., L'Hôpital's rule) to get a value of 0 for $x \in (0;1]$ and 1 for $x = 0$. Integrating this function gives a value of 0.

(b) The partial sums $s_n$ are monotone since $b_k$ and $r$ are nonnegative. The partial sums are also bounded, since $(b_k)$ is bounded (say, $b_k \le M$ for all $k$), and $r < 1$, so that

$$s_n \le M \sum_{k=1}^{n} r^k = \frac{Mr(1-r^n)}{1-r} \le \frac{Mr}{1-r}$$

Thus we have a bounded, monotone sequence of real numbers, so the Monotone Convergence Theorem says this sequence must converge. (Note that it need not converge to $Mr/(1-r)$, since $M$ was just a bound on $(b_k)$; rather, it converges to $r/(1-r)\limsup_k b_k$).

Problem 4:

(a) $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, and $H_3(x) = 8x^3 - 12x$.

(b) Follow the hint and let $v(x) = e^{-x^2}$, so the term in the hint is (where $v^{(m)}$ is the $m^{th}$ derivative of $v$)

$$\begin{aligned}
(-1)^n \int_R H_n(x)v^{(m)}\,dx &= H_n(x)v^{(m-1)}\Big|_R - \int_R 2nH_{n-1}(x)v^{(m-1)}\,dx \\
&= -\int_R 2nH_{n-1}(x)v^{(m-1)}\,dx \\
&= \ldots \\
&= (-1)^n 2^n n! \int_R H_0(x)v^{(m-n)}\,dx
\end{aligned}$$

and $H_0(x) = 1$. If $n < m$, integrating once more gives 0 since $v$ and its derivatives approach zero as $x$ goes to $\pm\infty$, and this proves the orthogonality.

(c) This follows directly from part (b), since we have just moved the weight function to $\varphi$.

(d) Because this is an orthonormal basis, we just calculate

$$f_8 = \int_R f(x)c_8\varphi_8(x)\,dx.$$

Problem 5:

(a) Let $0 \in \operatorname{int} C$ and $x \in X$. Then there is an $\epsilon > 0$ such that $B_\epsilon(0) \subseteq C$, and in particular $\frac{\epsilon}{2} \in C$, so $\rho_C(x) \le 2/\epsilon < \infty$. Now let $C$ be convex, and let $x, y \in C$ with $\rho_C(x) = \alpha$ and $\rho_C(y) = \beta$. Then $x' = x/\alpha \in C$ and $\in \operatorname{int}$

( x)      ( d) 8x 2 C, and thus the hyperplane de ned by f x 2 X : ( x) = ( d)g separatesd and C.