ßuid conduits have been observed experimentally [2,4], but their properties have never been studied.

scale L is proportional to the uniform conduit radius while vertical variations are assumed to be weak according to

$$\tilde{r} = r/L, \quad \tilde{z} = {}^{1/2} z/L, \quad L = R_0/\sqrt{8}. \tag{18}$$

The proportionality constant in the characteristic length L is chosen for convenience in working with the governing equations but will be rescaled to arrive at the standard form of the conduit equation (1). The boundary is now denoted by $r = (z,t) = R_0 + R(z,t)$ or $\tilde{r} = (R_0 + R(z,t))/L \quad \tilde{R}(\tilde{z},\tilde{t})$. Hence the unit normal and tangent vectors for the conduit are given by

$$\hat{n}_c = \frac{1}{\tilde{n}_c}\begin{pmatrix} -1 \\ {}^{1/2}\frac{\tilde{R}}{\tilde{z}} \end{pmatrix}, \quad \hat{t}_c = \frac{1}{\tilde{t}_c}\begin{pmatrix} {}^{1/2}\frac{\tilde{R}}{\tilde{z}} \\ 1 \end{pmatrix}, \tag{19a}$$

where

$$\tilde{n}_c = \tilde{t}_c = \left[1 + \left(\frac{\tilde{R}}{\tilde{z}}\right)^2\right]^{1/2}. \tag{20}$$

Velocities are normalized to the radially averaged vertical velocity of the uniform conduit

$$\tilde{u}^{(i,e)} = u^{(i,e)}/U, \quad U = \frac{gR_0^2(\ ^{(e)} - \ ^{(i)})}{8\mu^{(i)}}, \tag{21}$$

leading to the long time scale ${}^{-1/2}T$ for vertical dynamics where

$$\tilde{t} = {}^{1/2} t/T, \quad T = L/U. \tag{22}$$

To nondimensionalize the pressure, the characteristic scale is chosen so that the vertical pressure gradient within the conduit balances the viscous force due to radial variation in the vertical velocity,

$$\tilde{p}^{(i,e)} = {}^{1/2}\frac{p^{(i,e)} - p_0}{}, \quad = \mu^{(i)}U/L. \tag{23}$$

Like in dimensional variables, the nondimensional, modified pressure can be decomposed as $\tilde{p}^{(i,e)} = \tilde{P}^{(i,e)} - \tilde{p}_h^{(i,e)}$, where $\tilde{P}^{(i,e)} = {}^{1/2}P^{(i,e)}/$ is the scaled, absolute pressure and $\tilde{p}_h^{(i,e)}$ is the normalized hydrostatic pressure which takes the form

$$\tilde{p}_h^{(i,e)} = -\ {}^{1/2}\frac{^{(i,e)}gz}{} = \frac{-\ ^{(i,e)}\tilde{z}}{^{(e)} - \ ^{(i)}}. \tag{24}$$

Surface tension was neglected in the discussion of the uniform conduit, but it will be included in the full system of equations for completeness, so it is normalized about a characteristic scale :

$$\tilde{} = / . \tag{25}$$

The Reynolds numbers for the viscous fluid conduit system are therefore defined for the two fluids according to

$$Re^{(i,e)}$$

derive information about higher order corrections in special cases. In what follows, we determine the scalings such that all corrections to the conduit equation (1) are $O(\epsilon)$.

A. Viscous, higher order corrections

The equations solved in deriving the conduit equation (1) in Sec. III B were a special case of the Stokes' flow equations, in which the vertical dynamics occurred over a much longer length scale than the radial dynamics. A convenient analytical property of the axisymmetric Stokes' flow equations, is that one can rewrite the nondimensional equations in the form [23]

$$\tilde{\nabla}^2 p^{(i,e)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p^{(i,e)}}{\partial r}\right) + \epsilon^2\frac{\partial^2 p^{(i,e)}}{\partial z^2} = 0, \tag{58}$$

$$\tilde{\mathcal{L}}^2\psi^{(i,e)} = 0, \quad \tilde{\mathcal{L}} = \epsilon^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}, \tag{59}$$

where $\psi^{(i,e)}$ is the Stokes' stream function, which is related to the velocity components by

$$u_r^{(i,e)} = -\epsilon^{1/2}\frac{1}{r}\frac{\partial\psi^{(i,e)}}{\partial z}, \quad u_z^{(i,e)} = \frac{1}{r}\frac{\partial\psi^{(i,e)}}{\partial r}. \tag{60}$$

In the asymptotic formulation, the fluid pressures and velocities were expanded in asymptotic series and expressions for the leading order term in the expansion were found. It was unclear

[redacted]

FIG. 2. (Color online) Figure [redacted]
vertically uniform, intrusive cond [redacted]
then perturbed by a steplike incre [redacted]
One can see the formation of a DS [redacted]
wavetrain slowly modulated over [redacted]
interesting observation is that the [redacted]
within individual waves of the DSW [redacted]
ßuid.

and Whitehead [5] observed a simila [redacted]
the amplitude of conduit waves exce [redacted]
the small slope condition $^{1/2}$R 1 in [redacted]
evaluation gives $^{1/2}$R 4.5 and 2, respe [redacted]
indeed the perturbed conduit radius ex [redacted]
threshold and inertial effects would need to [redacted]
model the conduit dynamics accurately for la [redacted]
Hence our criteria accurately predict the point [redacted]
of the conduit equation as an approximate desc [redacted]
interface.

[redacted] With the limits of validity of the theoretic [redacted]
[redacted] blished, viscous ßuid conduits provide an [redacted]
[redacted] r the precise, quantitative, experimental st [redacted]
[redacted] ispersively regularized shock waves have a [redacted]
[redacted] eal of interest in recent years due to their o [redacted]
[redacted] ange of physical systems, to include ultraco [redacted]
[redacted] 26,27], ion-acoustic plasma [28], nonlinear op [redacted]
[redacted] nd shallow water [31], but careful compariso [redacted]
[redacted] nd data are lacking. One difÞculty is the lo [redacted]
[redacted] me scales required for the study of DSWs [redacted]
[redacted] odulated wavetrains are characterized by [redacted]
[redacted] o scales. One is the O (1) scale of individual os [redacted]
[redacted] e other is a long, slow scale of wave modu [redacted]
[redacted] 1 (generally, ÷is different from deÞned [redacted]
[redacted] wever, images from previous experiments o [redacted]
[redacted] experimental study of DSWs is access [redacted]
[redacted] d conduits, e.g., Fig. 2. In this setting, a DS [redacted]
[redacted] a steplike increase in the injection rate. [redacted]
[redacted] ger trailing, vertically uniform conduit c [redacted]
[redacted] er, leading vertically uniform conduit b [redacted]
[redacted] ating conduit waves, as depicted in Fig. 2 [redacted]
[redacted] tomated syringe pump, high resolutio [redacted]
[redacted] e measurement of the ßuid densities [redacted]
[redacted] quantitative experiments are possible [redacted]
[redacted] dity of the conduit equation (1), meas [redacted]
[redacted] stic DSW featuresÑleading and traili [redacted]
[redacted] g edge amplitudeÑcan be comp [redacted]
[redacted] sults of asymptotic modulation the [redacted]
[redacted] r the conduit equation (1) by the pre [redacted]

[redacted] equation is asymptotically [redacted]
[redacted] s in the small amplitude, lo [redacted]
[redacted] SW regimes, e.g., backß [redacted]
[redacted] n observed in large ampl [redacted]
[redacted] work shows that these f [redacted]
[redacted] c features of the red [redacted]
[redacted] in viscous ßuid con [redacted]

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