In implementing this procedure the sums or integrals involved in the inverse Radon transform of step

Here and throughout the paper we will consider the most singular term of the integral representations of the singly scattered fields. However, by making this approximation in the direct problem we are not making an additional approximation with respect to the inverse problem. This is because we proceed to solve the inverse problem *modulo a smooth error*. To obtain such a solution it is sufficient to consider only the most singular term in the direct problem. Indeed, only this term contributes to the most singular term that is recovered. This observation is discussed in greater detail in Appendix B.

To obtain the most singular term of the singly scattered field we consider the most singular part of the Green's function and its derivatives:

$$\tilde{G}(s,x,t)\approx\tilde{A}(s,x)\,\delta(t-\tilde{\phi}(s,x)),\qquad \hat{G}(x,r,t)\approx\hat{A}(x,r)\,\delta(t-\hat{\phi}(x,r)),\tag{1.11-1.12}$$

$$\tilde{G}_{,j}(s,x,t)\approx-\tilde{A}(s,x)\,\tilde{\phi}_{,j}(s,x)\,\delta'(t-\tilde{\phi}(s,x)),\tag{1.13}$$

$$\hat{G}_{,j}(x,r,t)\approx-\hat{A}(x,r)\,\hat{\phi}_{,j}(x,r)\,\delta'(t-\hat{\phi}(x,r)).\tag{1.14}$$

Substituting (1.11)-(1.14) in (1.10) we arrive at

$$U(s,r,t)=-\partial_t^2\int_D \mathrm{d}x\,[\kappa'+\sigma'\tilde{\phi}_{,j}\hat{\phi}_{,j}]\tilde{A}\hat{A}\,\delta(t-\tilde{\phi}-\hat{\phi}).\tag{1.15}$$

Here the phase functions $\tilde{\phi}=\tilde{\phi}(s,x)$ and $\hat{\phi}=\hat{\phi}(x,r)$ satisfy the eikonal equations with the background sound speed $c^0=\sqrt{\sigma^0/\kappa^0}$

So far our analysis has concerned the direct scattering problem. It is our goal, however, to consider (1.21) not as an expression for $U(s, r, t)$ but as an integral equation for f, where U is known. We recall that U is the scattered field and does not contain the unperturbed field due to the background material

Equation (2.14) for the phase functions simplifies and the phase functions $\tilde{\phi}^P$, $\hat{\phi}^P$, $\tilde{\phi}^S$ and $\hat{\phi}^S$ satisfy the eikonal equations

$$\sum_{j=1,2,3} [\tilde{\phi}^P_{,j}]^2 = \sum_{k=1,2,3} [\hat{\phi}^P_{,k}]^2 = \frac{1}{c_P^2}, \tag{2.27}$$

$$\sum_{j=1,2,3} [\tilde{\phi}^S_{,j}]^2 = \sum_{k=1,2,3} [\hat{\phi}^S_{,k}]^2 = \frac{1}{c_S^2}, \tag{2.28}$$

where $c_P(x) = \sqrt{(\lambda^0(x) + 2\mu^0(x))/\rho^0(x)}$ and $c_S(x) = \sqrt{\mu^0(x)/\rho^0(x)}$ are the P-wave speed and the S-wave speed. Here $\rho^0(x)$, $\lambda^0(x)$ and $\mu^0(x)$ are the density and elastic constants for the background isotropic elastic medium.

The amplitudes $\tilde{A}^P_{jl}$, $\hat{A}^P_{kl}$, $\tilde{A}^S_{jl}$ and $\hat{A}^S_{kl}$ satisfy transport equations which can be written as

$$(\rho^0 c_P^2 \tilde{A}^P_{jp} \tilde{A}^P_{jp} \tilde{\phi}^P_{,m})_{,m} = 0, \qquad \text{no summation over } j, \tag{2.29}$$

$$(\rho^0 c_P^2 \hat{A}^P_{kp} \hat{A}^P_{kp} \hat{\phi}^P_{,m})_{,m} = 0, \qquad \text{no summation over } k, \tag{2.30}$$

$$(\rho^0 c_S^2 \tilde{A}^S_{jp} \tilde{A}^S_{jp} \tilde{\phi}^S_{,m})_{,m} = 0, \qquad \text{no summation over } j, \tag{2.31}$$

$$(\rho^0 c_S^2 \hat{A}^S_{kp} \hat{A}^S_{kp} \hat{\phi}^S_{,m})_{,m} = 0, \qquad \text{no summation over } k. \tag{2.32}$$

We obtain the high frequency approximation of the integral representation of the singly scattered field by substituting (2.17)–(2.26) into (2.12). Then

$$U_{jk}(s, r, t) = U^{PP}_{jk}(s, r, t) + U^{PS}_{jk}(s, r, t) + U^{SP}_{jk}(s, r, t) + U^{SS}_{jk}(s, r, t), \tag{2.33}$$

where

$$U^{PP}_{jk} = -\partial_t^2 \int_D [\rho' \delta_{pl} + c'_{lmpq} \tilde{\phi}^P_{,q} \hat{\phi}^P_{,m}] \tilde{A}^P_{jp} \hat{A}^P_{kl}\, \delta(t - \tilde{\phi}^P - \hat{\phi}^P)\, dx, \tag{2.34}$$

$$U^{PS}_{jk} = -\partial_t^2 \int_D [\rho' \delta_{pl} + c'_{lmpq} \tilde{\phi}^P_{,q} \hat{\phi}^S_{,m}] \tilde{A}^P_{jp} \hat{A}^S_{kl}\, \delta(t - \tilde{\phi}^P - \hat{\phi}^S)\, dx, \tag{2.35}$$

$$U^{SP}_{jk} = -\partial_t^2 \int_D [\rho' \delta_{pl} + c'_{lmpq} \tilde{\phi}^S_{,q} \hat{\phi}^P_{,m}] \tilde{A}^S_{jp} \hat{A}^P_{kl}\, \delta(t - \tilde{\phi}^S - \hat{\phi}^P)\, dx, \tag{2.36}$$

$$U^{SS}_{jk} = -\partial_t^2 \int_D [\rho' \delta_{pl} + c'_{lmpq} \tilde{\phi}^S_{,q} \hat{\phi}^S_{,m}] \tilde{A}^S_{jp} \hat{A}^S_{kl}\, \delta(t - \tilde{\phi}^S - \hat{\phi}^S)\, dx. \tag{2.37}$$

We introduce the unit vectors tangent to the rays at the point x, denoting them by $\tilde{\alpha}^P = (\tilde{\alpha}^P_1, \tilde{\alpha}^P_2, \tilde{\alpha}^P_3)$, $\tilde{\alpha}^S = (\tilde{\alpha}^S_1, \tilde{\alpha}^S_2, \tilde{\alpha}^S_3)$, $\hat{\alpha}^P = (\hat{\alpha}^P_1, \hat{\alpha}^P_2, \hat{\alpha}^P_3)$ and $\hat{\alpha}^S = (\hat{\alpha}^S_1, \hat{\alpha}^S_2, \hat{\alpha}^S_3)$. Then

$$\tilde{\alpha}^P_j = c_P \tilde{\phi}^P_{,j}, \qquad \tilde{\alpha}^S_j = c_S \tilde{\phi}^S_{,j}, \qquad \hat{\alpha}^P_j = c_P \hat{\phi}^P_{,j}, \qquad \hat{\alpha}^S_j = c_S \hat{\phi}^S_{,j}, \tag{2.38–2.41}$$

where $j = 1, 2, 3$ and we have used (2.27) and (2.28). Let us introduce angles $\theta^{PP} = \theta^{PP}(x, s, r)$, $\theta^{PS} =$

$(\tilde{\alpha}_1^{SP})^2+(\tilde{\alpha}_2^{SP})^2=1$ and $\tilde{A}_j^{SP}$ satisfies the transport equation

$$(\rho^0 c_S^2 \tilde{A}_j^{SP} \tilde{A}_j^{SP} \tilde{\phi}^S_{,m})_{,m}=0, \qquad \text{no summation over } j. \tag{2.60}$$

We also have

$$\tilde{A}^S_{jp}\tilde{\beta}^{SP}_p=\tilde{A}^{SP}_j\tilde{\alpha}^{SP}_1 . \tag{2.61}$$

We rewrite the integrand of eq. (2.36) in a form similar to (2.55) to obtain

$$U^{SP}_{jk}=\partial_t^2\int_D \rho^0\left[\frac{\rho'}{\rho^0}\sin\theta^{SP}+\frac{\mu'}{\mu^0}\frac{c_S}{c_P}\sin 2\theta^{SP}\right]\tilde{A}^S_{jp}\tilde{\beta}^{SP}_p\hat{A}^P_k\,\delta(t-\tilde{\phi}^S-\hat{\phi}^P)\,\mathrm{d}x. \tag{2.62}$$

S-to-S scattering. In this case we need the unit vector $\gamma^{SS}=(\gamma_1^{SS},\gamma_2^{SS},\gamma_3^{SS})$ in the direction of $\tilde{\alpha}^S\times\hat{\alpha}^S$ as well as unit vectors $\hat{\beta}^{SS}=(\hat{\beta}_1^{SS},\hat{\beta}_2^{SS},\hat{\beta}_3^{SS})$ orthogonal to both $\hat{\alpha}^S$ and γ^{SS} and $\tilde{\beta}^{SS}=(\tilde{\beta}_1^{SS},\tilde{\beta}_2^{SS},\tilde{\beta}_3^{SS})$

The behavior of the operators $\mathbf{F}_m$ is determined by their phase and amplitude in the neighborhood of the set $C_\Phi = \{(\omega, s, r, x, x) \in R_+ \times \partial D \times \partial D \times D \times D\}$, the projection of which on $D \times D$ is the diagonal. See part 3 of Appendix B for more details. Our next step is to expand the phase of the Fourier integral [illegible]

[illegible]

of the Taylor series to obtain the most singular term of the operator F [illegible]

[illegible]

$$\phi(x, \tilde{\alpha}, \hat{\alpha}) - \phi(y, \tilde{\alpha}, \hat{\alpha}) \approx (\tilde{\phi}_{,j}(s, y) + \hat{\phi}_{,j}(y, r))(x_j - y_j)$$

$$= \left(\frac{1}{[illegible]}\tilde{\alpha}_i + \frac{1}{[illegible]}\hat{\alpha}_i\right)(x_i - y_i). \qquad (3.15)$$

[redacted]

We choose the weight $B(y, \tilde{\alpha}, \hat{\alpha})$ in (3.16) to be of the form

$$B(y, \tilde{\alpha}, \hat{\alpha}) = B(y, \tilde{\alpha} \cdot \hat{\alpha}) = \frac{1}{16 \operatorname{mes} E_\psi} \frac{1}{\cos \frac{1}{2}\theta} \left(\frac{\Delta^2 + 4 \cos^2 \frac{1}{2}\theta}{\tilde{c}(y)\, \hat{c}(y)} \right)^{3/2} b(y, \theta), \tag{3.21}$$

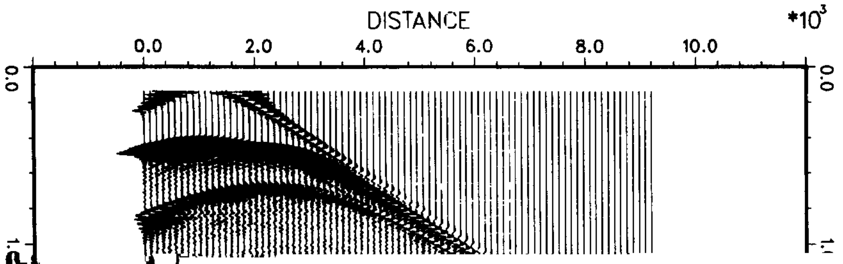
where $\operatorname{mes} E_\psi = \int_{E_\psi(\nu,\theta)} \mathrm{d}\psi$, and $b(y, \theta)$ is a function yet to be described.

Using the relation (Burridge and Beylkin, [30])

$$\mathrm{d}\tilde{\alpha}\, \mathrm{d}\hat{\alpha} = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\nu\, \mathrm{d}\psi, \tag{3.22}$$

(see Appendix C) and integrating over E_ψ we arrive at

[redacted]


DISTANCE
*10³
0.0
2.0
4.0
6.0
8.0
10.0
0.0
0.0

We compute for $m = 1, 2$

$$f_m^{\text{est}}(y) = \frac{1}{\pi^2 \rho^0(y)} \int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r \, B(y, s, r) \, J(y, s) \, J(y, r)$$

$$\cdot \cos^{2-m}\theta \cos^{m-1} 2\theta \frac{\tilde{A}_{jp}^{SS}(s, y) \, \tilde{\beta}_p^{SS} \, \hat{A}_{kl}^{S}(y, r) \, \hat{\beta}_l^{SS}}{\|\tilde{A}^{SS}(s, y)\|_1^2 \, \|\hat{A}^{SS}(y, r)\|_1^2} \, U_{jk}^{S^1 S^1}(s, r, t)\bigg|_{t=\phi(y,s,r)}, \tag{5.24}$$

where

$$\|\hat{A}^{SS}\|_1^2 = \sum_{k=1,2,3} (\hat{A}_{kl}^{S} \hat{\beta}_l^{SS})^2, \qquad \|\tilde{A}^{SS}\|_1^2 = \sum_{j=1,2,3} (\tilde{A}_{jp}^{S} \tilde{\beta}_p^{SS})^2, \tag{5.25–5.26}$$

and

$$B(y, s, r) = \frac{b(y, \theta) \cos^2 \frac{1}{2}\theta}{2 \operatorname{mes} E_\psi \, [c_S(y)]^3}. \tag{5.27}$$

We also compute

$$q_{lm}(y) = \int b(y, \theta) \sin \tfrac{1}{2}\theta \cos^{l+m-2} 2\theta \cos^{4-l-m} \theta \, \mathrm{d}\theta, \quad l, m = 1, 2. \tag{5.28}$$

Solving the system (3.35) with a_{lm} given by (5.36) and f_m^{est} by (5.33) yields the perturbations of the parameters in (5.29).

6. The Kirchhoff approximation

[illegible]

where θ_s is the angle between the normal and the ray connecting the source with the point x. At the specular point (6.3) becomes

$$R(x, \tfrac{1}{2}\theta) = f(x, \theta)/4\cos^2\tfrac{1}{2}\theta, \tag{6.4}$$

where θ is the angle between the rays connecting source and receiver to the point x.

We also observe that at the specular point

$$\mathrm{N}_x \cdot \nabla\phi(x, s, r) = |\nabla\phi(x, s, r)| = (2\cos\tfrac{1}{2}\theta)/c(x), \tag{6.5}$$

where $c(x)$ is the velocity.

We now rewrite (6.1) as a volume integral using the singular function $\gamma(x)$ of the surface:

$$U(s, r, \omega) = \mathrm{i}\omega \int \mathrm{d}x\, \gamma(x)\, R(x, \theta_r)\, \tilde{A}(s, x)\, \hat{A}(x, r)\, \mathrm{N}_x \cdot \nabla\phi(x, s, r)\, \mathrm{e}^{\mathrm{i}\omega\phi(x,s,r)}. \tag{6.6}$$

[redacted]

[redacted]

can see that an extra factor $\mathrm{i}\omega$ is present in (1.21) whereas an extra angular dependent factor is present

Following the derivation in Section 3 we introduce the new system of coordinates described there. We choose weight $B(y, \tilde{\alpha}, \hat{\alpha})$ as follows

$$B(y, \tilde{\alpha}, \hat{\alpha}) = \frac{1}{\text{mes } E_\psi} \frac{\cos^3 \frac{1}{2}\theta}{c^2} b(y, \theta). \tag{6.11}$$

We note that it is exactly the same weight that is described in (3.37) with $d = 1$ and $\Delta = 0$. Also we note that $d = 1$ in (3.37) corresponds to multiplying the scattered field by $\mathrm{i}\omega$ and that is exactly what we have done for the Kirchhoff approximation in (6.8).

Following the derivation in Section 3 we arrive at

$$(\mathrm{F}_m^0 f)(y) = \frac{1}{(2\pi)^3} \int_D \mathrm{d}x \int_0^\infty \omega^2 \,\mathrm{d}\omega \int_{E_\theta} \mathrm{d}\theta \int_{S_\nu} \sin \tfrac{1}{2}\theta \frac{\mathrm{N}_y \cdot \nabla\phi(y, \tilde{\alpha}, \hat{\alpha})}{4 \cos^2 \theta_s}$$

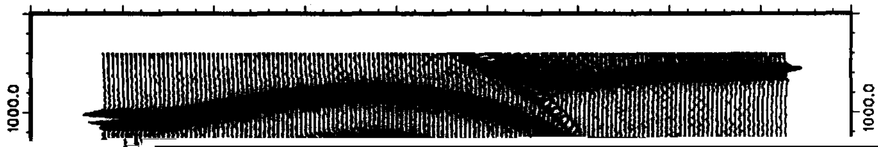
$$16 \cos^4 \tfrac{1}{2}\theta \; [illegible]$$

where $m = 1, 2, 3$.

At this point we make use of the fact that $\mathrm{WF}(\gamma) = (x, \xi_x)$, where $\xi_x \neq 0$ is any vector in the direction of N_x, which exists because we assume that the surface is smooth so that it has a normal at every point (see Appendix D for definitions and explanations).

For every point of $\mathrm{WF}(\gamma)$ we construct a conic neighborhood (see Appendix D) and, to isolate this conic neighborhood, we define a neutralizer (or cut-off function) which is C^∞, equal to one in the conic neighborhood, and equal to zero outside some strip surrounding the conic neighborhood. We split the operator (6.12) into the sum of two operators, one with the amplitude unchanged, and another with the amplitude equal to zero in the conic neighborhood of $\mathrm{WF}(\gamma)$. The latter operator is regularizing in

DISTANCE



and the worst

$$\langle f_1(y)\rangle - 2\langle f_2(y)\rangle + \langle f_3(y)\rangle.$$

[redacted]

for the first combination,

$$\mathrm{d}\log(\lambda+2\mu) - 4\frac{c_S^2}{c_P^2}\,\mathrm{d}\log\mu \approx \frac{\lambda'}{\lambda^0+2\mu^0} - \frac{2\mu'}{\lambda^0+2\mu^0}, \tag{7.10}$$

for the second, and

$$\mathrm{d}\log(\lambda+2\mu) - 2\mathrm{d}\log\rho \approx \frac{\lambda'}{\lambda^0+2\mu^0} - 2\frac{\rho'}{\rho^0} + \frac{2\mu'}{\lambda^0+2\mu^0} \tag{7.11}$$

for the third

Appendix B

Part 1

The argument that it is sufficient to consider the leading order term in the forward problem to account for the most singular term in the linearized inverse problem is presented in [40] for the reduced wave equation. We describe here the main points of this argument.

[redacted]

frequency domain as

$$U(s, r, \omega) = -\int_D \mathrm{d}x\,[\omega^2 \kappa' \tilde{G}\hat{G} - \sigma' \tilde{G}_{,j}\hat{G}_{,j}]. \tag{B.1}$$

We then construct the operator

$$(\mathbf{R}_m^* U)(y) = -\frac{1}{\pi^2}\int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r \int_{-\infty}^{+\infty} \mathrm{d}\omega \frac{\bar{\tilde{G}}(s, y, \omega)\,\bar{\hat{G}}(y, r, \omega)}{|\tilde{G}|^2\,|\hat{G}|^2}\, h_m(s, r, y)\, U(s, r, \omega), \tag{B.2}$$

with $m = 1, 2$.

In this formula $\bar{\tilde{G}}$ and $\bar{\hat{G}}$ are complex conjugates of $\tilde{G}$ and $\hat{G}$ and $h_m(s, r, y)$ is yet to be defined. If [redacted]

Proof. Let ψ be a unit vector perpendicular to ν and lying in the plane of ξ, η. Choose coordinates so that ν lies along the x_n axis, and ξ, η, ν, ψ lie in the (x_{n-1}, x_n)-plane. Then from (C.2) it is clear that

$$\mathrm{d}\xi_k = \cos\alpha\,\mathrm{d}\nu_k - \sin\alpha\,\mathrm{d}\psi_k \quad \text{for } k = 1, \ldots, n-2, \tag{C.3}$$

[redacted]

and so

$$\mathrm{d}\xi_k\,\mathrm{d}\eta_k = \sin\theta\,\mathrm{d}\nu_k\,\mathrm{d}\psi_k, \quad \text{for } k = 1, \ldots, n-2. \tag{C.4}$$

Here we have used (C.3) and the addition formula for sine.

[redacted]

those vectors. Then

$$\mathrm{d}\xi' = \mathrm{d}\nu' - \mathrm{d}\alpha, \qquad \mathrm{d}\eta' = \mathrm{d}\nu' + \mathrm{d}\beta, \tag{C.5}$$

and so

$$\mathrm{d}\xi'\,\mathrm{d}\eta' = \mathrm{d}\nu'(\mathrm{d}\alpha + \mathrm{d}\beta) = \mathrm{d}\nu'\,\mathrm{d}\theta, \tag{C.6}$$

Thus, on combining (C.4) and (C.6) we have

[redacted]

[redacted]

Definition 2. A distribution u is C^∞ in the conic open subset of $\Omega \times (R^n \backslash \{0\})$ if it is C^∞ in a conic neighborhood of every point of the subset.

Definition 3. The complement in $\Omega \times (R^n \backslash \{0\})$ of the union of all conic open sets in which the distribution is C^∞ is called the wave-front set $\mathrm{WF}(u)$ of the distribution u.

[redacted]

$$(\mathbf{A}u)(x) = \int \mathrm{d}\xi \, a(x, \xi) \, \hat{u}(\xi) \, \mathrm{e}^{\mathrm{i}\xi \cdot x}, \tag{D.2}$$

where $a(x, \xi)$ is the standard symbol of the operator $\mathbf{A}$.

[redacted]

w_l	(3.1)	weighting functions
$x=(x_1, x_2, x_3)$	(1.1)	cartesian position vector
$\alpha=(\alpha_1, \alpha_2, \alpha_3)$	(2.38)	unit tangent to ray
$\beta=(\beta_1, \beta_2, \beta_3)$	(2.50)	unit vector perpendicular to ray
$\Delta(y)$	(3.19)	dimensionless parameter
δ	(1.4)	Dirac delta
δ_{ij}	(1.4)	Kronecker delta
$\theta=\theta(x, s, r)$	(1.19)	angle between rays at x
$\kappa(x)$	(1.1)	compressibility
λ	(2.2)	Lamé constant
μ	(2.2)	Lamé constant
$\rho=\rho(x)$	(2.1)	density
$\sigma(x)$	(1.1)	specific volume
$\nu=(\nu_1, \nu_2, \nu_3)$	(3.18)	unit vector at point of reflection
$\hat{\phi}(s, x)$	(1.12)	phase of $\hat{G}$
$\tilde{\phi}(s, x)$	(1.11)	phase of $\tilde{G}$
$\phi(x, s, r)=\tilde{\phi}(s, x)+\hat{\phi}(x, r)$	(3.5)	two-way travel time (phase)
ω	(3.11)	angular frequency
Ω_θ	(3.29)	domain of integration
∂_t	(1.1)	derivative with respect to t
$*_t$	(1.9)	convolution in t
^{0}	(1.2)	pertaining to the background

[9] W. Chang, P. Carrion and G. Beylkin, "Wave front sets of solutions to linearized inverse scattering problems", *Inverse Problems 3*, 683–690 (1987).
[10] J.G. Hagedoorn, "A process of seismic reflection interpretation", *Geophysical Prospecting 2*, 85–127 (1954).
[11] G.H.F. Gardner, ed., *Migration of Seismic Data*, Geophysics reprints series, no. 4, Society of Exploration Geophysics (1985).
[12] R.H. Stolt and A.B. Weglein, "Migration and inversion of seismic data", *Geophysics 50*, 2458–2472 (1985).
[13] I.M. Gel'fand and B.M. Levitan, "On the determination of a differential equation from its spectral function", *Izv. Akad. Nauk SSSR 15*, 309 (1951) (*also*: *Amer. Math. Soc. Transl. 1*, 253–304 (1955).
[14] [illegible]