On the Design of Highly Accurate and Efficient IIR and FIR Filters

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***Abstract*—We describe a systematic method for designing highly accurate and efficient infinite impulse response (IIR) and finite impulse response (FIR) filters given their specifications. In our approach, we first meet the specifications by constructing an IIR filter with, possibly, a large number of poles. We then construct, for any given accuracy, an optimal IIR version of such filter (with a minimal number of poles). Finally, also for any given accuracy, we convert the IIR filter to an efficient FIR filter cascade (either serial or parallel). Since in this FIR approximation the non-causal part of the IIR filter only introduces an additional delay (as a function of the desired accuracy), our IIR construction does not have to enforce causality. Thus, we obtain a simple method for constructing linear phase filters if the specifications so require. All of these procedures are accomplished via robust, fast algorithms. We provide several illustrative examples of our method.**

***Index Terms*—Approximation algorithms, digital filter design, FIR filters, IIR filters, optimal rational approximations, quadrature mirror filters.**

I. Introduction

IN HIS 2006 paper "The Rise and Fall of Recursive Digital Filters," [1] Rader gives a brief history of filter design response $h_d(n)$, up to some finite but arbitrary accuracy $\varepsilon > 0$ over a certain range of the index $n \quad \mathbb{Z}$.

Our solution makes use of an algorithm in [2], [9]. Given a sequence

$$h_d(n), \quad 1 \leq n \leq 2N + 1$$

and a target accuracy $\varepsilon > 0$, we determine the optimal (minimal) number of nodes γ_m and weights w_m such that

$$\left| h_d(n) - \sum_{m=1}^{M} w_m \gamma_m^n \right| < \varepsilon, \quad 1 \leq n \leq 2N + 1. \tag{II.2}$$

We now describe the steps of the algorithm to obtain this approximation.

Algorithm 1:

- Build the $N + 1 \times N + 1$ Hankel matrix

$$\mathbf{H}_{k\ell} = h_d(k + \ell + 1 \;\; 1 \;\; 1$$

Remark 2. In many cases of practical interest, some type of symmetry exists between $h_d(n)$ and $h_d(-n)$. In such cases a corresponding symmetry is induced between poles inside and outside the unit disk and their corresponding weights. For example, it is quite common for the impulse response to be real and symmetric,

$$h_d(n) \quad \mathbb{R} \quad \text{and} \quad h_d(-n) = h_d(n),$$

in which case it is not difficult to show that poles appear at conjugate-reciprocal locations and the corresponding weights are complex conjugates, so that with a suitable reordering

this process introduces some approximation error, so we will allocate a portion of the total allowable error, as given in the filter specifications, to each of the three steps.

Consider the following lowpass filter specification:

$$\begin{aligned} \left|H(e^{j\omega}) - 1\right| &< 10^{-4}, \quad |\omega| < \frac{80}{140} \\ \left|H(e^{j\omega})\right| &< 10^{-4}, \quad |\omega| > \frac{81}{140}, \end{aligned} \tag{III.1}$$

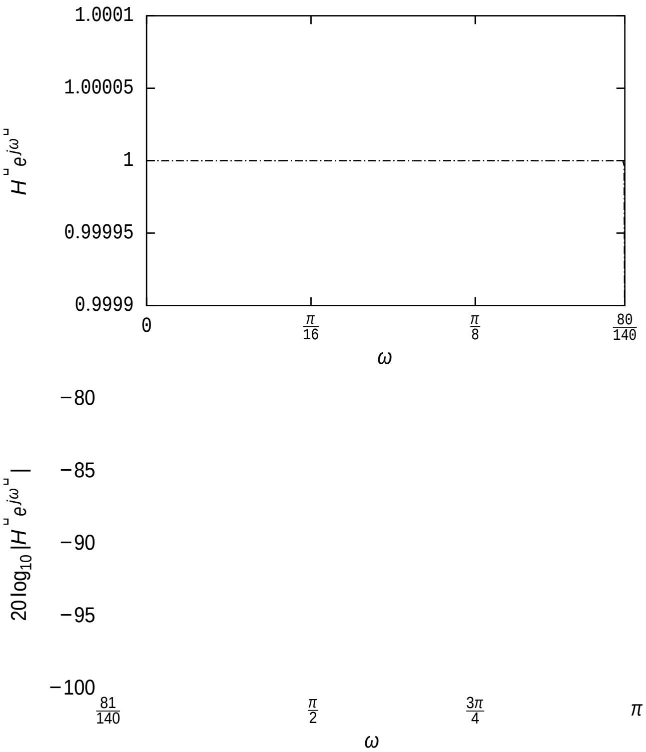

Figure III.1. Frequency response of the lowpass filters H_d (dash-dot line), H (solid line), and $\tilde{H}$ (dashed line) in the passband (top) and the stopband (bottom). $\tilde{H}$ is an excellent approximation of H

produces serious numerical difficulties effectively precluding yields fast codes. For hardware-based realizations, the serial cascade structure in [5] may also be considered.

Remark 12. A non-causal filter lacking a symmetric impulse response does not possess symmetry of poles inside and outside the unit disk. In this situation, the poles inside the unit disk may be applied using the standard recursive equations and the poles outside the unit disk using an appropriate FIR

Table II
POLES AND WEIGHTS OF THE "STAIRCASE" FILTER H IN SECTION V-A, AND THE NUMBER OF FACTORS REQUIRED FOR EACH POLE IN $\hat{H}$. THE CONSTANT TERM IS $w_0 = 0.60057$.

Pole z_m	Weight w_m	Factors
$0.73729 + 0.64330j$	$2.6451\text{e-}5 - 3.7663\text{e-}3j$	9
$-0.21254 + 0.94461j$	$7.2305\text{e-}6 - 5.5944\text{e-}3j$	8
$0.67809 + 0.59327j$	$-4.3178\text{e-}4 - 1.1810\text{e-}2j$	7
$-0.18571 + 0.83540j$	$6.8318\text{e-}4 - 1.7861\text{e-}2j$	6
$0.44780 + 0.42670j$	$-1.3985\text{e-}2 - 7.0914\text{e-}2j$	4
$-8.7442\text{e-}2 + 0.42165j$	$-2.9205\text{e-}2 - 0.16703j$	4
-0.50671	$-1.4759\text{e-}2$	3

Such filters give rise to filter banks, and, with simple additional constraints, to orthonormal wavelet bases. Filter banks provide methods for efficiently applying operators to signals, in particular, operators that in the standard representation result in very long filters, such as fractional derivatives or the Hilbert transform (see, e.g., [22]). Filter banks have proven useful for applications in signal processing, numerical analysis, and data compression (see, e.g., [23]).

Depending on the application, we may request different properties of the filter (V.1). Algebraically, many of these properties are interrelated and several are mutually exclusive. For example, no FIR QMF can be symmetric but nothing prevents the design of symmetric IIR QMFs. We note that many such restrictions on properties of QMFs are fragile; i.e., for any finite accuracy these restrictions disappear, and we use this fact as a tool for the design of approximate QMFs with the desired properties. Some examples may be found in [5] and here we construct approximate IIR and FIR QMFs that are symmetric (i.e., have linear phase), efficient, and have attractive flatness and subband isolation properties.

In [24] a particularly interesting family of symmetric IIR QMFs is introduced,

$$E_{4N}(z) = \frac{(1+z)^{2N}\left((1+z)^{2N} + (-1)^N\sqrt{2}(1-z)^{2N}\right)}{(1+z)^{4N} + (1-z)^{4N} + (-1)^N\sqrt{2}(1-z^2)^{2N}}, \quad \text{(V.2)}$$

where the positive integer parameter N simultaneously controls the flatness of the passband and stopband and the width of the transition region. It may be that the value N required to achieve a sufficiently narrow transition band results in a filter that is excessively flat. We show how to use our method to obtain an efficient FIR approximation of the original QMF that retains the desired sharpness but gains efficiency by reducing the excessive flatness. An example of such a QMF frequency response is illustrated in Fig. V.3.

The filter flatness is controlled by the root of order $2N$ at $z = -1$ of $E_{4N}(z)$. To obtain a more efficient, but less flat, IIR filter, we factor our a portion of this high-order root and apply the reduction algorithm from Section II-B to the remaining terms. Observing that $E_{4N}(z)$ is real-valued on the unit circle, we select some integer $S < N$ (which controls the flatness of the new filter) and rewrite $E_{4N}(z)$ as

$$E_{4N}(z) = \left(\frac{1+z}{2}\right)^S \left(\frac{1+z^{-1}}{2}\right)^S \times \left(c + \sum_{n=1}^{2N} \frac{s_n}{1 - p_n/z} + \bar{s}_n \right.$$

Table III
POLES, WEIGHTS, AND NUMBER OF FACTORS REQUIRED IN THE FIR APPROXIMATION $\tilde{E}_{80}$ OF THE IIR QMF E_{80} IN SECTION V-B. THE CONSTANT TERM IS $w_0 = -1.24533870508$.

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