NAME: _____

SECTION: 001 at 9:05 am

Instructions:

- 1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
- 2. Calculators are permitted.
- 3. Justify your answers, show all work.
- 4.

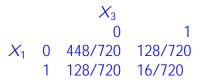
(c) Suppose the number of foreign fragments in a portion of peanut butter is a random variable with a mean of 3 and a variance of 3. Suppose a random sample of 50 portions of peanut butter are collected and the average number of foreign fragments in a portion is calculated. What is the probability of observing an average of 3.6 or more fragments?

Solution:

- (a) (10 points)
 - (i) (5 points) The possible outcomes of X_1 and X_3 are:

Outcome	X_1	X_3	Prob.
YYG	1	0	2/10 * 1/9 * 8/8 = 16/720
YGY	1	1	2/10 * 8/9 * 1/8 = 16/720
YGG	1	0	2/10 * 8/9 * 7/8 = 112/720
GYY	0	1	8/10 * 2/9 * 1/8 = 16/720
GYG	0	0	8/10 * 2/9 * 7/8 = 112/720
GGY	0	1	8/10 * 7/9 * 2/8 = 112/720
GGG	0	0	8/10 * 7/9 * 6/8 = 336/720

The joint pmf of X_1 and X_3 is:



The joint pmf of X_1 and X_3 after simpli cation is:

$$\begin{array}{ccc} & X_3 \\ & 0 & 1 \\ X_1 & 0 & 28/45 & 8/45 \\ & 1 & 8/45 & 1/45 \end{array}$$

(c) (6 points) By the Central Limit Theorem, $\overline{X}^{approx} N(3; 2 = \frac{3}{50})$. P(X > 3:6) = P(Z > $\frac{3:6-3}{:245})$ P(Z > 2:45) = 1 (2:45) 1 .9929 = .0071:

Problem 2. (28 points) Let (X; Y) be jointly distributed random variables with conditional pdf given by:

$$f_{YjX}(yjx) = \begin{pmatrix} \frac{2y}{x^2} & 0 < y < x \\ 0 & \text{otherwise} \end{pmatrix}$$

and marginal pdf of X given by:

$$f_X(x) = \begin{cases} 5x^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Find the joint pdf of X and Y.
- (b) Find the marginal pdf of Y:
- (c) Find E[Y|X] and use it to nd the expectation of Y:
- (d) Find Cov(X; Y).

Solution:

(a) (7 points)

$$f_{X|Y}(x|y) = f_{YJX}(y|x) f_X(x)$$
(
$$f_{X|Y}(x|y) = \begin{cases} 10x^2y & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (7 points)

$$f_{Y}(y) = \int_{y}^{Z} 10x^{2}y \, dx$$
$$= \frac{10x^{3}y}{3} \int_{y}^{1}$$
$$= \frac{10y}{3} \frac{10y^{4}}{3}$$
$$f_{Y}(y) = \int_{0}^{1} \frac{10y}{3} \frac{10y^{4}}{3} \quad 0 < y < 1$$
otherwise

Problem 3. (18 points) If X and

(b) Suppose that the time it takes (in hours) to replace a failed battery is uniformly distributed over (0, .8), independently. Approximate the probability that all batteries have failed before 1000 hours have passed.

Solution:

(a) (11 points)

Let X_i be the lifetime of the *i*th battery, i = 1/2/2200: $E[X_i] = 4/2 Var(X_i) = 16$ By the CLT, $\sum_{n=1}^{200} X_i^{approx} N(800/3200)$

$$P(\sum_{n=1}^{\infty} X_i > 810) \quad P \quad Z > \frac{810}{P} \frac{800}{3200}$$
$$= P(Z > :1768)$$
$$1 \quad (:18)$$
$$= 1 \quad :5714 = :4286$$

(b) (15 points)

Let R_i be the time needed to replace the *ith* battery, i = 1/2/2/200: E $[R_i] = .4$; Var $(R_i) = .8^2$ 1=12 = .0533

$$E\begin{bmatrix} P_{n=1}^{199} R_i \end{bmatrix} = 79.6; \text{ Var} \begin{pmatrix} P_{n=1}^{199} R_i \end{pmatrix} = 10.6067$$

By the CLT, $P_{n=1}^{200} X_i + P_{n=1}^{199} R_i^{approx} N(879.6; 3210.61)$
$$P\begin{pmatrix} \times 0 \\ n=1 \end{pmatrix} R_i < 1000 P Z < \frac{1000}{P} \frac{879.6}{3210.61}$$
$$= P(Z < 2.1249)$$
$$(2.12)$$
$$= .983$$

Bonus Problem. (3 points) Let X be a random variable with mean 50 and variance 25. If X is the number of cars produced in a week at a particular auto manufacturing plant, nd a lower bound on the probability that the production of cars in a week is between 30 and 70.

Solution: P X 50 20 $\frac{Var(X)}{400} = \frac{25}{400} = \frac{1}{16}$. P X 50 < 20 1 $\frac{1}{16} = \frac{15}{16}$.