Write your name below. This exam is worth 100 points. On each problem (except for problem 1),

2. (19 points) Consider the following matrix A

$$A = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 1 & 2 & -3 & 0 \\ 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 \end{bmatrix}.$$

(a) (7 points) Find the permutation matrix P such that $B := PA$ is symmetric. Show both P and B.

(b) (12 points) Can B be factored as LDL^T? If yes, find the factorization. If no, justify why it cannot be factored.

Solutions:

(a) Note that the last three rows already look symmetric. Then we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}.$$

(b) Note that since B is symmetric, it could be possible. According to the theorem, we need to know if B is regular. We attempt to perform LU factorization, first:

3. (20 points: 10 each)

The following two problems are unrelated.

(a) Determine if the following matrices are linearly independent

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix}.$$

(b) Let $V = \mathbb{R}^4$ and $W \subseteq V$ be the space spanned by the vectors: $\begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -3 \\ -5 \end{pmatrix}.$

and the dimension of W is two.

(Solution 2) Another solution (albeit slightly harder) is to consider the matrix whose columns are the vectors and find it REF:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 8 \\ 5 & 1 & -3 \\ -3 & -4 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & -9 & -18 \\ 0 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here we are looking for the column space so we see that columns one and two are pivot columns and so select the pivot columns from the original matrix A:

$$\text{basis for } W = (1; -2; 5; -3)^T; (2; 3; 1; -4)^T$$

4. (19 points)

The following two questions are unrelated.

(a) (9 points) Let $V = \mathbb{R}^3$ and $W = \left\{ (x, y, z)^T \in V : x^2 - 2xy + y^2 - z^2 = 0 \right\}$. Is W a vector subspace of V? Prove or disprove.

(b) (10 points) Consider $F(I)$, the vector space of real valued functions on an interval I. Do the solutions to the differential equation

$$y'' + 5y' + 2y = 0$$

form a subspace of $F(I)$? Prove that they do or show that they do not.

Solution:

(a) W is not a vector space as it is not closed under vector addition:

Let $w_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ be members of W, then

$w_1 + w_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$, which is not a member of W.

(b) This is a subspace. It is non-empty ($y = 0$ is a solution) and closed under both scalar multiplication and vector addition:

For any $c \in \mathbb{R}$ we have
$(cy)'' + 5(cy)' + 2(cy) = c(y)'' + 5c(y)' + 2cy$

$= c(y'' + 5y' + 2y) = c(0) = 0.$

For any 2 solutions to the differential equation we have
$(y_1 + y_2)'' + 5(y_1 + y_2)' + 2(y_1 + y_2) = y_1'' + y_2'' + 5y_1' + 5y_2' + 2y_1 + 2y_2$

$= (y_1'' + 5y_1' + 2y_1) + (y_2'' + 5y_2' + 2y_2) = 0$

5.