1. (32 pts) Suppose the density of the surface $z = 1 - x^2$ is $\rho = |xy|$ g/cm^2 and consider the vector field

$$\mathbf{F} = \langle 3x + \cos y, 2y + \sin z, e^x + 5z \rangle$$

   (a) Find the mass of the part of the surface lying above the region in the xy-plane between $y = 0$ and $y = 2$.

   (b) Find the outward flux of $\mathbf{F}$ through the **closed** surface enclosing the region below $z = 1 - x^2$, above the xy-plane and between $y = 0$ and $y = 2$.

Solution:

(a)

$$g(x, y, z) = x^2 + z \implies \nabla g = \langle 2x, 0, 1 \rangle \implies \|\nabla g\| = \sqrt{4x^2 + 1}$$

Project surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$, integration region $-1 \le x \le 1, 0 \le y \le 2$ and $|\nabla g \cdot \mathbf{p}| = 1$

$$\begin{aligned} \text{Mass} &= \int_{-1}^{1}\int_{0}^{2} |xy| \sqrt{4x^2 + 1}\, dy\, dx \quad \text{(integrand even in } x \text{ and separable)} \\ &= 2\int_0^1 x\sqrt{4x^2 + 1}\, dx \int_0^2 y\, dy \\ &= \frac{1}{4}\int_1^5 u^{1/2}\, du \left[\frac{y^2}{2}\right]_0^2 = \frac{1}{3}\left(5\sqrt{5} - 1\right) \end{aligned}$$

(b) The surface S and the region W it encloses satisfy the hypotheses of Gauss' (Divergence) Theorem with

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(3x + \cos y) + \frac{\partial}{\partial y}(2y + \sin z) + \frac{\partial}{\partial z}(e^x + 5z) = 3 + 2 + 5 = 10$$

$$\begin{aligned} \text{Flux} &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F}\, dV = \int_{-1}^{1}\int_0^2\int_0^{1-x^2} 10\, dz\, dy\, dx \\ &= 10\int_{-1}^{1}\int_0^2 \left(1 - x^2\right) dy\, dx = 20\int_{-1}^{1}\left(1 - x^2\right) dx = 40\int_0^1 \left(1 - x^2\right) dx = \frac{80}{3} \end{aligned}$$

2. (16 pts) Find the area under the graph of $z = 100(x^2 + 2y^2)$ lying above the second quadrant portion of the **curve** $x^2 + y^2 = 4$.

Solution: The area is given by $\int_C f(x, y)\, ds$ where $f(x, y) = 100(x^2 + 2y^2)$. C can be parameterized by

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \ \pi/2 \le t \le \pi \implies \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} \implies \|\mathbf{r}'(t)\| = 2$$

Thus

$$\begin{aligned} \text{Area} &= \int_C 100(x^2 + 2y^2)\, ds = \int_{\pi/2}^{\pi} 100\left(4\cos^2 t + 8\sin^2 t\right)(2)\, dt = 800\int_{\pi/2}^{\pi}\left(\cos^2 t + 2\sin^2 t\right) dt \\ &= 800\int_{\pi/2}^{\pi}\left(1 + \sin^2 t\right) dt = 800\int_{\pi/2}^{\pi}\left(\frac{3}{2} - \frac{1}{2}\cos 2t\right) dt = 400\left[3t + \frac{1}{2}\sin 2t\right]_{\pi/2}^{\pi} = 600\pi \end{aligned}$$

3. (16 pts) I am doing laps around the unit circle (counterclockwise) in the presence of the force field

$$\mathbf{F} = \langle Axy - By^3, 4y + 3x^2 - 3xy^2 \rangle$$

(a) After having gone from $(1; 0)$

(This was easily obtained since we knew where the plane intersects the coordinate axes. A point and two vectors in the plane could also have been used to find the plane's equation).

To obtain the orientation of the surface induced by the orientation of its boundary requires the use of ∇g. Projecting the surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$ and $|\nabla g \cdot \mathbf{p}| = 1$ with the area of integration

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1; 0 \le y \le 2 - 2x\}$$

Note that the surface could have been projected onto the xz- or yz-plane.

We need the curl of $\mathbf{V}$, given as

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \cos^2 \sqrt{z} \end{vmatrix} = (x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$$

Then

$$\begin{aligned} \text{Circulation} &= \oint_C \mathbf{V} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{V} \cdot d\mathbf{S} = \iint_R (\nabla \times \mathbf{V}) \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|}\, dA \\ &= \iint_R [(x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}] \cdot \frac{(-2\mathbf{i} - \mathbf{j} - \mathbf{k})}{1}\, dA \\ &= \int_0^1 \int_0^{-2x+2} (-2x + 3y - 1)\, d \end{aligned}$$

END OF EXAM