1. (40 pts) Let $g(x, y) = x^3 - 3xy + y^3$.

   (a) Find and classify the critical points of $g(x, y)$.

   (b) Find the maximum rate of change of $g(x, y)$ at the point $(2, 1)$ and the direction in which it occurs.

   (c) The origin and the point $(2, 1, 3)$ lie on the surface $z = g(x, y)$. Find an equation for the plane that passes through the points and contains the line with symmetric equations $x = \frac{y}{3} = z$.

   (d) Starting at the origin, a fly takes off from the surface $z = g(x, y)$ and travels along the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 7t^2\mathbf{k}$, $t \geq 0$. At what value(s) of t will the fly meet the surface again?

Solution:

(a)

$$\begin{aligned} g(x, y) &= x^3 - 3xy + y^3 \\ g_x &= 3x^2 - 3y \\ g_y &= -3x + 3y^2 \end{aligned}$$

The critical points occur where $g_x = 0$ and $g_y = 0$.

$$\begin{aligned} g_x &= 3x^2 - 3y = 0 \implies y = x^2 \\ g_y &= -3x + 3y^2 = 0 \implies -3x + 3x^4 = 0 \implies x = 0, 1 \end{aligned}$$

There are two critical points at $(0, 0)$ and $(1, 1)$. Apply the Second Derivative Test.

$$g_{xx} = 6x \qquad g_{yy} = 6y \qquad g_{xy} = -3$$

$$\begin{aligned} D(x, y) &= g_{xx}g_{yy} - (g_{xy})^2 \\ D(0, 0) &= 0 \cdot 0 - (-3)^2 = -9 < 0 \\ D(1, 1) &= 6 \cdot 6 - (-3)^2 = 27 > 0 \quad \text{and} \quad g_{xx}(1, 1) = 6 > 0 \end{aligned}$$

Therefore there is a saddle point at $g(0, 0) = 0$ and a local minimum at $g(1, 1) = -1$.

(b)

$$\nabla g(x, y) = \langle 3x^2 - 3y, -3x + 3y^2 \rangle$$

The gradient vector $\nabla g(2, 1) = \langle 9, -3 \rangle$ is the direction of maximum rate of change, and the maximum rate is

$$|\nabla g(2, 1)| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}.$$

(c) Let $v_1 = \langle 2, 1, 3 \rangle$ be the vector connecting the two points and let $v_2 = \langle 1, 3, 1 \rangle$ be the direction vector of the line. Then a normal vector to the plane is

$$v_1 \times v_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix} = -8\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

and an equation of the plane is $-8x + y + 5z = 0$.

(d) Substituting $x = t$, $y = t$, and $z = 7t^2$ into $z = g(x; y$[redacted]

$$7t^2 = t^3 \quad 3t^2 + t^3 \implies 10t^2 =$$ [redacted] $t = 0; 5.$

The fly begins on the surface at $t = 0$ and meets the su[redacted] at $t = 5$.

2. (15 pts) Consider the integral

$$\int_0^3 \int_{1-x}^{1+x} \frac{x \quad y}{x + y} \, dy$$ [redacted]

Use the transformation $u = x \quad y$, $v = x + y$ to set up an [redacted]ntegral over a reg[redacted]v plane. Sketch both the xy and uv regions. Do not evaluate [redacted].

Solution:

Letting $u = x \quad y$ and $v = x + y$ gives $x = {}^{u+v}$

An equivalent integral over the uv-plane is

$$\int_{-1}^{5}\int_{1}^{6-u}\frac{1}{2}\frac{u}{v}\,dv\,du \qquad \text{or} \qquad \int_{1}^{7}\int_{-1}^{6-v}\frac{1}{2}\frac{u}{v}\,du\,dv.$$

3. (25 pts) The volume of a solid is given in cylindrical coordinates by $\displaystyle\int_{\theta=\pi/2}^{\pi}\int_{0}^{6}\int_{r}^{6} r\,dz\,dr\,d\theta$.

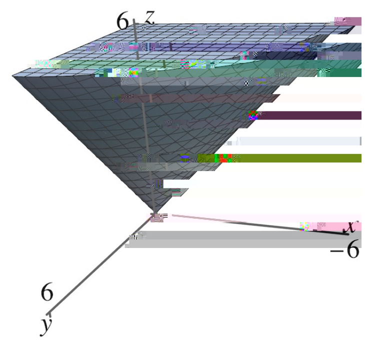
   (a) Sketch and shade the 2D cross-sections of the solid in the rz-plane (for a constant θ) and in the xy-plane. Label all intercepts.

   (b) Set up (but do not evaluate) an equivalent integral in rectangular coordinates in the order $dz\,dy\,dx$.

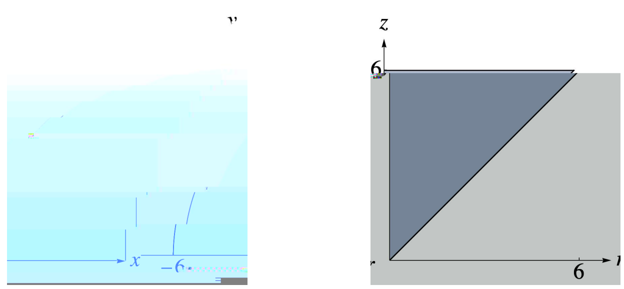
   (c) Set up (but do not evaluate) an equivalent integral in spherical coordinates in the order $d\rho\,d\phi\,d\theta$.

Solution:

The solid is a quarter cone above the second quadrant of the xy-plane, bounded below by $z = r = \sqrt{x^2+y^2}$ and above by the plane $z = 6$.



(a)



(b) In rectangular coordinates, an equation for the cone is $z = \sqrt{x^2+y^2}$. A semicircle of radius 6 centered at the origin has the equation $y = \sqrt{36 - x^2}$

4. (25 pts)

(a) Use Gaussian elimination to solve the linear system.

$$\begin{aligned} 2x + 4y \quad &= 10 \\ x \quad 4y + z &= 6 \\ x + y \quad &= 4 \end{aligned}$$

(b) Reduce this homogeneous system to RREF and use the result to find the complete solution set.

$$\begin{aligned} 2x + 4y \quad &= 0 \\ x \quad 4y + z &= 0 \end{aligned}$$

Solution:

(a) First row reduce the augmented matrix

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & 1 \\ 0 & 0 & 1 & & & \end{array}\right.$$

7.