2. (15 pts) Suppose we have the series

$$s = \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3^2}{2\cdot 4}\right) + \ln\left(\frac{4^2}{3\cdot 5}\right) + \ln\left(\frac{5^2}{4\cdot 6}\right) + \cdots$$

(a) Find a simple expression for the partial sums s_n of the series s.

(b) Does the series converge or diverge? Fully justify your answer. If the series converges, find its sum.

Solution:

(a) To get a handle on the partial sum, let's look at the first few partial sums:

$$s_1 = \ln\left(\frac{2}{3}\right)$$

$$s_2 = \ln\left(\frac{2}{}\right.$$

3. (15 pts) Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{4}{(n!)^2}$.

   (a) Show that the series converges.

   (b) Estimate the error in using the partial sum s_3 to approximate s.

Solution:

(a) The series is alternating with $b_n = \frac{4}{(n!)^2}$. Now, clearly b_n is decreasing as the denominator grows with n. Further, $\lim_{n\to\infty} b_n = \frac{1}{\infty} = 0$. So by the Alternating Series Test, the1

4. (25 pts) Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n!}(n-1)!$$

(a) Find the radius and interval of convergence for the power series $f(x)$

(b) Using interval notation, for what values of x is $f(x)$ absolutely convergent, conditionally convergent, and divergent?

Solution:

(a) Before we use any tests, we can simplify the factorials in the series as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n!}(n-1)! = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n(n-1)!}(n-1)! = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n}$$

Next, applying the Ratio Test yields

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{4(n+1)} \frac{4n}{(-1)^n (x-2)^n}\right| = \left|\frac{(x-2)^n (x-2)}{(n+1)} \frac{n}{(x-2)^n}\right| = \frac{n}{n+1}|x-2|$$

which gives a limit of

$$L = \lim_{n\to\infty} \frac{n}{n+1}|x-2| = |x-2|.$$

From the Ratio Test, we must have

$$L = |x-2| < 1 \quad = \quad -1 < x-2 < 1 \quad = \quad 1 < x < 3$$

for convergence (possibly including the endpoints). This inequality implies $\boxed{R = 1.}$ Further, we almost have the full interval of convergence but we need to check the endpoints. For the left endpoint when $x = 1$, we have

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-2)^n}{4n} = \sum_{n=0}^{\infty} \frac{1}{4n}$$

which is the divergent Harmonic Series multiplied by 1/4. Moving to the right endpoint when $x = 3$, we have

$$f(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (3-2)^n}{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n}$$

which is the convergent Alternating Harmonic Series multiplied by 1/4. Putting everything together, we find that the interval of convergence is $\boxed{(1, 2].}$

(b) From part (a), we know the series is

$$\boxed{\textit{divergent for } (-\infty, 1] \cup (3, \infty).}$$

Further, the right endpoint gave the Alternating Harmonic Series which is conditionally convergent since the absolute value of the series yields the regular divergent Harmonic Series. As a result, $f(x)$ is

$$\boxed{\textit{conditionally convergent for } \{3\}.}$$

Lastly, the Ratio Test gives absolute convergence so the from the $L < 1$ inequality, we have

$$\boxed{\textit{absolute convergence for } (1, 3).}$$

5. (15 pts) Starting with the Maclaurin series for $\frac{1}{1-x}$, write out a power series for the function below and determine its radius of convergence without the use of the Ratio or Root Tests.

$$f(x) = \frac{5}{1-4x^2}.$$

Solution: We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$. Now to find our series, we just need to replace x with $4x^2$ in our geometric series and multiply by 5 to get

$$f(x) = \frac{5}{1-4x^2} = 5\sum_{n=0}^{\infty}(4x^2)^n = \boxed{\sum_{n=0}^{\infty} 5 \cdot 4^n x^{2n}.}$$

To find the radius of convergence we replace x with $4x^2$ in the original convergence criteria to get

$$\left|4x^2\right| < 1 \quad = \quad \left|x^2\right| < \frac{1}{4} \quad = \quad |x| \le \frac{1}{2}$$

meaning the radius of convergence is $\boxed{R = \frac{1}{2}.}$

Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$